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DEMAZURE ROOTS AND SPHERICAL VARIETIES: THE EXAMPLE OF HORIZONTAL SL_2 -ACTIONS

KEVIN LANGLOIS AND ALEXANDER PEREPECHKO

ABSTRACT. Let G be a connected reductive group, and let X be an affine G -spherical variety. We show that the classification of \mathbb{G}_a -actions on X normalized by G can be reduced to the description of quasi-affine homogeneous spaces under the action of a semi-direct product $\mathbb{G}_a \rtimes G$ with the following property. The induced G -action is spherical and the complement of the open orbit is either empty or a G -orbit of codimension one. These homogeneous spaces are parametrized by a subset $\mathrm{Rt}(X)$ of the character lattice $\mathbb{X}(G)$ of G , which we call the set of Demazure roots of X . We give a complete description of the set $\mathrm{Rt}(X)$ when G is a semi-direct product of SL_2 and an algebraic torus; we show particularly that $\mathrm{Rt}(X)$ can be obtained explicitly as the intersection of a finite union of polyhedra in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}(G)$ and a sublattice of $\mathbb{X}(G)$. We conjecture that $\mathrm{Rt}(X)$ can be described in a similar combinatorial way for an arbitrary affine spherical variety X .

Key words: Demazure root, Luna–Vust theory, polyhedral divisor, spherical variety.

MSC 2010: 14M25 14M27 14R20.

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INTRODUCTION

In this article we are interested in the automorphism groups of spherical varieties defined over an algebraically closed field \mathbb{K} of characteristic zero. Let G be a connected reductive group. Recall that a G -spherical variety is a normal variety endowed with

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a G -action and containing an open orbit of a Borel subgroup. The Luna–Vust theory describes such varieties with combinatorial objects defined by the geometric structure of the open G -orbit [Kno91]. Their study has been motivated by numerous important examples (see [Sat60, Dem70, KKMS73, MO73, Pau81, CP83]) coming in particular from representation theory and enumerative geometry.

To begin studying automorphisms of G -spherical varieties it is natural to consider the varieties with a relatively small automorphism group, for instance which neutral component is expected to be generated by the G -action. The following families of such spherical varieties are studied:

- (1) smooth complete toric varieties [Dem70, AG10];
- (2) flag varieties [Dem77];
- (3) spherical regular varieties [BB96];
- (4) wonderful varieties [Bri07, Pez09];

On the other side, there are families of spherical varieties with quite rich automorphism group, namely such that their automorphism group acts infinitely transitively on the regular locus. Excluding one-dimensional case, the following families are appropriate:

- (5) affine cones over flag varieties [AKZ12, Section 1];
- (6) non-degenerate affine toric varieties [AKZ12, Section 2];
- (7) equivariant SL_2 -embeddings considered as spherical varieties under the extension of SL_2 by a torus [BH08], [AFKKZ13, Section 5.2];
- (8) smooth affine G -spherical varieties under a semisimple group G [AFKKZ13, Section 5.2];

Related to this topic, our general aim is to provide a method of constructing automorphisms of an arbitrary G -spherical variety in a combinatorial way. In this article we study the classification of the \mathbb{G}_a -actions on affine G -spherical varieties that are normalized by G . Restricting to toric varieties, such actions allow studying the automorphism groups in case (1) via total coordinate spaces [Cox95, Cox14]. They also allow establishing the flexibility [AKZ12, Definition 0.1] in case (6).

In order to explain our results, we introduce the following notation. Let X be an affine G -spherical variety. We say that two \mathbb{G}_a -actions $\varphi_1 : \mathbb{G}_a \times X \rightarrow X$ and $\varphi_2 : \mathbb{G}_a \times X \rightarrow X$ are *equivalent*, if there exists a constant $\lambda \in \mathbb{K} \setminus \{0\}$ such that

$$\varphi_1(\lambda\mu, x) = \varphi_2(\mu, x) \text{ for any } x \in X, \mu \in \mathbb{G}_a.$$

In particular, two equivalent actions share the same orbits. Denote the G -action on X by $(g, x) \mapsto g \cdot x$. A \mathbb{G}_a -action

$$\gamma : \mathbb{G}_a \times X \rightarrow X, (\lambda, x) \mapsto \lambda * x$$

is called *normalized* by G if there exists a character $\chi : G \rightarrow \mathbb{G}_m$ such that

$$g \cdot (\lambda * (g^{-1} \cdot x)) = (\lambda \chi^{-1}(g)) * x$$

for all $x \in X$, $g \in G$, and $\lambda \in \mathbb{G}_a$. The \mathbb{G}_a -action normalized by G naturally defines an action of the semidirect product $\mathbb{G}_a \rtimes_\chi G$ on X , see 2.6.

In Theorem 4.5 we provide a correspondence in terms of Luna–Vust theory between

- equivalence classes of \mathbb{G}_a -actions on X normalized by G and
- quasi-affine homogeneous spaces under the action of a semi-direct product $\mathbb{G}_a \rtimes_\chi G$ enjoying the following property: the induced G -action is spherical and the complement of the open orbit is either empty or a G -orbit of codimension

one. Furthermore, the character χ satisfies a certain explicit condition imposed by X ; such characters are called *Demazure roots* of X and form the set denoted by $\text{Rt}(X)$. This definition is coherent with a classical one for toric varieties, see [Dem70, Section 4.5, p. 571].

The set $\text{Rt}(X)$ is defined in a geometric way. A natural question arises whether $\text{Rt}(X)$ is a combinatorial object. More precisely:

Question 0.1 (see 4.11). Can $\text{Rt}(X)$ be obtained explicitly as the intersection of a sublattice of the character lattice $\mathbb{X}(G)$ and a finite union of polyhedra in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}(G)$?

The answer to this question is affirmative if G is an algebraic torus. The simplest non-abelian connected reductive group which we can consider is the semidirect product $G_e = \text{SL}_2 \rtimes_e \mathbb{T}$. It is the reductive group defined via the multiplication law

$$(A, t_1) \cdot (B, t_2) = (A \cdot \varepsilon_e(t_1)(B), t_1 \cdot t_2), \text{ where } \varepsilon_e(t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \chi^{-e}(t)b \\ \chi^e(t)c & d \end{pmatrix}.$$

We prove that the answer is still affirmative for every affine G_e -spherical variety X , see Theorem 4.18. We also provide a necessary and sufficient condition for affine G_e -spherical homogeneous spaces to admit a nontrivial normalized \mathbb{G}_a -action, see 4.14.

To obtain these results we use *AL-colored polyhedral divisors* (by Arzhantsev and Liendo) which provide a combinatorial description of compatible SL_2 -actions on affine \mathbb{T} -varieties of complexity at most one, see [AL12]. This description allows us to characterize all spherical G_e -actions on affine varieties. In fact, we establish an explicit relation between AL-colorings on G_e -spherical varieties and corresponding colored cones in the Luna–Vust theory, see Theorem 3.7 and Propositions 3.13, 3.18.

Let us outline the structure of this article. In Section 1 we recall elements of the Luna–Vust theory for the embeddings of spherical homogeneous spaces. In Section 2 we briefly explain the Altmann–Hausen construction of polyhedral divisors (see [FZ03], [AH06], [Tim08]) for \mathbb{T} -varieties of complexity one and provide several known results concerning normalized \mathbb{G}_a -actions (from [FZ05], [Lie10a], [Lie10b], [AL12], [Lan13, Chapter 5], and [LL]). In Section 3 we establish the connection between AL-colored polyhedral divisors and colored cones. In Section 4 we deduce Theorem 4.5, we pursue Question 4.11, in particular we answer it for affine G_e -spherical varieties in Theorem 4.18, and we study the existence of normalized \mathbb{G}_a -actions in Theorem 4.14.

Convention 0.2. Throughout this paper \mathbb{K} is an algebraically closed field of characteristic zero. By a variety we mean an integral separated scheme of finite type over \mathbb{K} . If X is a variety then $\mathbb{K}[X]$ denotes the coordinate ring of X , and $\mathbb{K}(X)$ denotes its field of rational functions. A point of X is assumed to be a \mathbb{K} -rational point. Furthermore, all algebraic group actions are in particular morphisms of varieties. For an algebraic group G acting on X the natural action on $\mathbb{K}[X]$ (resp. $\mathbb{K}(X)$) is defined by the formula

$$(g \cdot f)(x) = f(g^{-1} \cdot x),$$

where $g \in G$, $f \in \mathbb{K}[X]$ (resp. $f \in \mathbb{K}(X)$), and $x \in X$.

1. PRELIMINARIES ON LUNA–VUST THEORY

In this section, we recall some notation from the Luna–Vust theory for the embeddings of spherical homogeneous spaces. We are mainly interested in the case where

the embeddings are affine. We refer the reader to [LV83, Kno91, Pez10, Tim11] for the general theory. Let G be a connected reductive linear algebraic group and consider a Borel subgroup $B \subset G$. We start by introducing classical definitions.

Definition 1.1. A variety with a G -action is called *spherical* (or G -spherical when one needs to emphasize the acting group) if it is normal and contains an open B -orbit.

To define the notion of embeddings of a spherical homogeneous space, we introduce the scheme of geometric localities (see [LV83, Section 1]).

1.2. Let G/H be a spherical homogeneous space. A subalgebra of $\mathbb{K}(G/H)$ is *affine* if it is finitely generated and if $\mathbb{K}(G/H)$ is equal to its field of fractions. A *geometric locality* is a local algebra obtained by the localization of an affine subalgebra of $\mathbb{K}(G/H)$ and having residue field equal to \mathbb{K} . The set $\text{Sch}(G/H)$ of geometric localities is naturally endowed with a structure of \mathbb{K} -scheme, where the spectrum of each affine subalgebra of $\mathbb{K}(G/H)$ is an open subset. The group G acts on $\text{Sch}(G/H)$ as an abstract group; this action is induced by the G -action on $\mathbb{K}(G/H)$. We denote by $\text{Emb}(G/H)$ the maximal G -stable normal open subset which the G -action is regular (see [LV83, Propositions 1.2 and 1.4]). An *embedding* of G/H is a G -stable separated noetherian open subset of $\text{Emb}(G/H)$. If G is an algebraic torus, then an embedding of G is called a *toric variety*.

The toric varieties are described by objects from the convex geometry called fans, e.g. see [KKMS73, MO73, Oda88, Ful93]. The Luna–Vust theory is a generalization of this description for embeddings of homogeneous spaces. In the sequel, we denote by X an affine embedding of a spherical homogeneous space G/H . We choose $x \in X$ so that $B.x$ is the open B -orbit.

Definition 1.3. Let K be an algebraic group acting on a variety Z . A K -divisor of Z is a K -stable prime divisor on Z . In our situation, a B -divisor of the spherical variety X that is not G -stable is called a *color*.

Identifying $G.x$ with G/H we may view a color as the closure in X of a B -divisor of G/H . Thus, colors are determined by the homogeneous space G/H .

1.4. The algebra of rational functions $\mathbb{K}(X) = \mathbb{K}(G/H)$ is endowed with a natural linear B -action. Consider the lattice L of B -weights of $\mathbb{K}(X)$ for this action. For a B -eigenvector f in $\mathbb{K}(X)$ we denote by χ_f the corresponding weight. Since the quotient of two eigenvectors of the same weight is B -invariant and therefore constant on the open orbit $B.x$, the B -eigenspaces in $\mathbb{K}(X)$ are of dimension one, hence the weight χ_f defines f up to a scalar. Summarizing, we have the exact sequence

$$1 \rightarrow \mathbb{K}^* \rightarrow \mathbb{K}(X)^{(B)} \rightarrow L \rightarrow 0,$$

where $\mathbb{K}(X)^{(B)}$ denotes the multiplicative subgroup of B -eigenvalues of $\mathbb{K}(X)^*$. The epimorphism $\mathbb{K}(X)^{(B)} \rightarrow L$ is given by $f \mapsto \chi_f$.

1.5. Recall that a *discrete valuation* on $\mathbb{K}(X)$ is a map $v : \mathbb{K}(X)^* \rightarrow \mathbb{Q}$ such that

- $v(f + g) \geq \min\{v(f), v(g)\}$ for all $f, g \in \mathbb{K}(X)^*$ satisfying $f + g \in \mathbb{K}(X)^*$;
- v is a group morphism from $(\mathbb{K}(X)^*, \times)$ to $(\mathbb{Q}, +)$;
- the subgroup \mathbb{K}^* is contained in the kernel of the morphism v .

Let $V = \text{Hom}(L, \mathbb{Z})$ be the dual lattice of L and let $V_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} V$ be the associated \mathbb{Q} -vector space. For every discrete valuation v of $\mathbb{K}(X)$ we associate a vector $\varrho(v) \in V_{\mathbb{Q}}$ by letting

$$\varrho(v)(\chi_f) = \langle \chi_f, \varrho(v) \rangle = v(f) \text{ for all } f \in \mathbb{K}(X)^{(B)}.$$

Since X is normal, each prime divisor $D \subset X$ naturally defines a discrete valuation v_D on $\mathbb{K}(X)$. So, we may identify any subset of B -divisors with the corresponding set of discrete valuations, which we denote by the same letter.

1.6. A discrete valuation v on $\mathbb{K}(X)$ is said to be *G-invariant* if every fiber of the map v is G -stable, for the natural linear G -action on $\mathbb{K}(X)$. It is well known that the restriction of ϱ to the subset of discrete G -invariant valuations is injective [LV83, Section 7.4]. Furthermore, the image of this restriction is a polyhedral cone \mathcal{V} of $V_{\mathbb{Q}}$ (see [BP87, Corollary 3.2]), called the *valuation cone* of G/H . Thus, we will say that V is the *valuation lattice*. Recall that the spherical variety X is said to be *horospherical* if the isotropy group of a general point contains a maximal unipotent subgroup of G . It is well-known that X is horospherical if and only $\mathcal{V} = V_{\mathbb{Q}}$ (see [Kno91, Corollary 6.2]).

Let us introduce the classical notion of colored cone [LV83, p. 230]. In this definition, an element of L is seen as a linear form on $V_{\mathbb{Q}}$.

Definition 1.7. A *colored cone* of the homogeneous space G/H is a pair $(\mathcal{C}, \mathcal{F})$, where

- \mathcal{F} is a subset of colors of G/H such that $0 \notin \varrho(\mathcal{F})$;
- $\mathcal{C} \subset V_{\mathbb{Q}}$ is a strongly convex polyhedral cone generated by the union of $\varrho(\mathcal{F})$ and of a finite subset of \mathcal{V} ;
- the intersection of the relative interior of \mathcal{C} with the cone \mathcal{V} is non-empty.

Moreover, a colored cone $(\mathcal{C}, \mathcal{F})$ will be called *affine* if there exists a lattice vector $m \in L$ satisfying the following conditions:

- m is non-positive on \mathcal{V} ;
- m is zero on \mathcal{C} ;
- m is positive on all colors of G/H not belonging to \mathcal{F} .

There is a natural way to build an affine colored cone of G/H from the embedding X , see [Kno91, Section 2]. In the next paragraph, we explain this construction.

1.8. Note that the open orbit $B.x$ is affine (see [Tim11, Theorem 3.5]), and so its complement in X is a union of B -divisors. So, X has a finite number of B -divisors. We denote by \mathcal{P} the set of G -divisors of X identified with a subset of $V_{\mathbb{Q}}$. Moreover, since X is affine, X has a unique closed G -orbit Y , which is contained in the closure of all the G -orbits of X . We denote by \mathcal{F}_Y the set of colors of X containing Y . The *associated colored cone* of X is the pair $(\mathcal{C}, \mathcal{F}_Y)$, where $\mathcal{C} \subset V_{\mathbb{Q}}$ is the cone generated by the union of \mathcal{P} and of the subset $\varrho(\mathcal{F}_Y)$. According to [Kno91, Section 2] and [Kno91, Theorem 6.7] the pair $(\mathcal{C}, \mathcal{F}_Y)$ is in fact an affine colored cone of G/H . Note that the embedding X is called *toroidal* if $\mathcal{F}_Y = \emptyset$.

Conversely, to a colored cone $(\mathcal{C}, \mathcal{F})$ of G/H one can associate an embedding.

1.9. Let $(\mathcal{C}, \mathcal{F})$ be a colored cone of G/H . For a discrete valuation v of $\mathbb{K}(G/H)$ we write \mathcal{O}_v the corresponding local algebra. Let X_B be the open B -orbit in G/H .

Considering a finite subset $\mathcal{P} \subset \mathcal{V}$ such that $\mathcal{P} \cup \varrho(\mathcal{F})$ generates the cone \mathcal{C} , we define the algebra

$$A(\mathcal{P}, \mathcal{F}) = \mathbb{K}[X_B] \cap \bigcap_{v \in \mathcal{P}} \mathcal{O}_v \cap \bigcap_{D \in \mathcal{F}} \mathcal{O}_{v_D}.$$

By [LV83, Section 8], $A(\mathcal{P}, \mathcal{F})$ is a normal affine subalgebra of $\mathbb{K}(G/H)$. The variety $X_0 = \text{Spec } A(\mathcal{P}, \mathcal{F})$ does not depend on the choice of the set \mathcal{P} . Furthermore, X_0 is an open subset of $\text{Emb}(G/H)$ (compare with [LV83, Proposition 8.10]). The associated spherical embedding of $(\mathcal{C}, \mathcal{F})$ is the subscheme $X = G \cdot X_0$ [LV83, Proposition 1.5]. This construction is also explained in [Tim11, Chapter 3, Section 13].

More precisely, we have the following classical result (see [Kno91, Theorem 6.7]).

Theorem 1.10. *The map defined by sending an affine embedding X of G/H to its associated colored cone $(\mathcal{C}, \mathcal{F}_X)$ is a bijection between*

- *the set of affine embeddings of G/H and*
- *the set of affine colored cones of G/H .*

The next well-known lemma will be useful afterwards (see the proof of [Kno91, 6.7]).

Lemma 1.11. *Let X be an affine embedding of G/H with associated colored cone $(\mathcal{C}, \mathcal{F}_X)$. Let \mathcal{F}_0 be the set of colors of G/H . Then the cone Γ dual to the cone generated by the B -weights of $\mathbb{K}[X]$ is the polyhedral cone in $V_{\mathbb{Q}}$ generated by the subset $\mathcal{C} \cup \varrho(\mathcal{F}_0)$. In addition, Γ is strongly convex.*

Proof. Let U be the unipotent part of B . Since X is affine, the field of fractions of $\mathbb{K}[X]^U$ is equal to $\mathbb{K}(X)^U$ (see [Tim11, Lemma D.7]). Consequently, every B -eigenvector of $\mathbb{K}(X)$ is the quotient of two eigenvectors of $\mathbb{K}[X]$. This shows that Γ is a strongly convex cone. The rest of the proof is straightforward. \square

2. PRELIMINARIES ON POLYHEDRAL DIVISORS AND NORMALIZED \mathbb{G}_a -ACTIONS

In this section, we recall some basic facts from the theory of polyhedral divisors for the case of complexity one [FZ03, AH06, Tim08]. We present also a brief survey of known results concerning the classification of normalized \mathbb{G}_a -actions on affine \mathbb{T} -varieties of complexity one, see [FZ05], [Lie10a], [Lie10b], [AL12], [Lan13, Chapter 5], and [LL].

2.1. Polyhedral divisors and affine \mathbb{T} -varieties of complexity one.

We denote by \mathbb{T} an algebraic torus of dimension n . We fix a lattice M isomorphic to the character group of \mathbb{T} via $m \mapsto \chi^m$.

Definition 2.1. The *complexity* of a \mathbb{T} -action on a variety X is the transcendence degree over \mathbb{K} of the field extension $\mathbb{K}(X)^{\mathbb{T}}$. By a result of Rosenlicht [Ros63], the complexity is also the codimension of a general orbit in X . In particular, if the action is faithful, then the complexity is equal to $\dim X - \dim \mathbb{T}$. A \mathbb{T} -variety is a normal variety endowed with a faithful \mathbb{T} -action.

Note that having a \mathbb{T} -action on an affine variety $X = \text{Spec } A$ is equivalent to endowing A with an M -grading. If \mathbb{T} acts on X and $m \in M$, then we define the m -th graded component of A by letting

$$A_m = \{f \in A \mid t \cdot f = \chi^m(t)f \text{ for all } t \in \mathbb{T}\}.$$

In [AH06], a combinatorial description of M -graded algebras corresponding to affine \mathbb{T} -varieties is given in terms of polyhedral divisors. We recall this construction in the setting of the complexity one. We also introduce other notation from convex geometry which will be useful in the sequel.

2.2. Let $N = \text{Hom}(M, \mathbb{Z})$ be the dual lattice of M . Denote by $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$ and $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ the associated vector spaces. For a polyhedral cone $\tau \subset N_{\mathbb{Q}}$ the symbols τ^{\vee} , $\text{lin.}(\tau)$, and $\text{rel.int.}(\tau)$ stand respectively for the dual cone in $M_{\mathbb{Q}}$, the linear part $(-\tau) \cap \tau$, and the relative interior of τ . Considering $v \in N_{\mathbb{Q}}$ we may write $v^{\perp} = \{m \in M_{\mathbb{Q}} \mid v(m) = 0\}$ for its orthogonal space and denote $\mu(v) = \inf\{r \in \mathbb{Z}_{>0} \mid rv \in N\}$.

For every polyhedron Δ of $N_{\mathbb{Q}}$ we denote by $\Delta(r)$ the set of its r -dimensional faces. Let $\sigma \subset N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. A subset $\Delta \subset N_{\mathbb{Q}}$ is called a σ -polyhedron if Δ is a Minkowski sum $Q + \sigma$ for a polytope¹ $Q \subset N_{\mathbb{Q}}$.

A σ -polyhedral divisor over a curve C is a formal sum $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$, where each Δ_z is a σ -polyhedron, with the condition that $\Delta_z = \sigma$ for all but finitely many $z \in C$. The finite subset $\{z \in C \mid \Delta_z \neq \sigma\}$ is called the *support* of \mathfrak{D} . The *degree* of \mathfrak{D} , denoted by $\deg \mathfrak{D}$, is the Minkowski sum of all the coefficients of \mathfrak{D} , which is a σ -polyhedron (see [AH06, 2.12]). For a vector $m \in M_{\mathbb{Q}}$ we let

$$\mathfrak{D}(m) = \sum_{z \in C} \min_{v \in \Delta_z(0)} \{v(m)\} \cdot z,$$

which is a \mathbb{Q} -divisor on C . If C is a normal curve, then the associated M -graded algebra of \mathfrak{D} is

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma^{\vee} \cap M} H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])) \chi^m.$$

The multiplication on $A[C, \mathfrak{D}]$ is defined on homogeneous elements via the maps

$$H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])) \chi^m \times H^0(C, \mathcal{O}_C([\mathfrak{D}(m')])) \chi^{m'} \rightarrow H^0(C, \mathcal{O}_C([\mathfrak{D}(m + m')])) \chi^{m+m'}$$

sending (f_1, f_2) to $f_1 \cdot f_2$. The polyhedral divisor \mathfrak{D} is called *linear* on a polyhedral cone $\omega \subset M_{\mathbb{Q}}$ if

$$\mathfrak{D}(m + m') = \mathfrak{D}(m) + \mathfrak{D}(m') \text{ for all } m, m' \in \omega.$$

We denote by $\Lambda(\mathfrak{D})$ the coarsest quasifan of σ^{\vee} where \mathfrak{D} is linear on each cone (see [AH06, Section 1] for details).

Definition 2.3. A σ -polyhedral divisor \mathfrak{D} on a normal curve C is said to be *proper* if $\mathfrak{D}(m)$ is semi-ample for any $m \in \sigma^{\vee}$ and big for any $m \in \text{rel.int.}(\sigma^{\vee})$. This condition is automatically fulfilled in case of an affine curve C . Otherwise, if C is projective, then the properness condition can be split in two parts:

- $\deg \mathfrak{D} \not\subset \sigma$;
- for any $m \in \sigma^{\vee}$ such that $\min_{v \in \deg \mathfrak{D}} \{v(m)\} = 0$, m belongs to the boundary of σ^{\vee} and $\mathfrak{D}(m)$ is a nonzero \mathbb{Q} -principal divisor.

In this paper we deal with proper polyhedral divisors over \mathbb{A}^1 or \mathbb{P}^1 . In this case, we have a simple characterization of the properness: a σ -polyhedral divisor over \mathbb{P}^1 is proper if and only if $\deg \mathfrak{D} \not\subset \sigma$.

¹We recall that a polytope in a \mathbb{Q} -vector space is the convex hull of a non-empty finite subset.

2.4. If \mathfrak{D} is a proper polyhedral divisor over a normal curve C , then $X = \operatorname{Spec} A[C, \mathfrak{D}]$ is an affine \mathbb{T} -variety of complexity one. The cone σ^\vee is the weight cone of the M -graded algebra $\mathbb{K}[X]$. The curve C is a rational quotient of X , i.e. $\mathbb{K}(X)^\mathbb{T} = \mathbb{K}(C)$. Conversely, every affine \mathbb{T} -variety of complexity one is obtained in this way [AH06, Theorem 3.1]. For the uniqueness of such representation, see [AH06, Section 8].

Let \mathfrak{D} be a proper σ -polyhedral divisor over a normal curve C . Consider the associated algebra $A = A[C, \mathfrak{D}]$ and let $X = \operatorname{Spec} A$. In the following paragraph we explain a classification of \mathbb{T} -divisors in terms of the pair (C, \mathfrak{D}) corresponding to X . See [AH06, Section 7], [Tim08], and [PS11] for further details.

2.5. Let $\sigma(1)^*$ be the following subset of $\sigma(1)$. If C is affine, then we let $\sigma(1)^* = \sigma(1)$. Otherwise, C is projective and $\sigma(1)^*$ is the set of 1-dimensional faces of σ that do not meet $\deg \mathfrak{D}$. For every element $\rho \in \sigma(1)$ we denote by the same letter ρ its extremal ray. There are two types of \mathbb{T} -divisors in the variety X .

- *Horizontal* divisors denoted by D_ρ for each $\rho \in \sigma(1)^*$. The corresponding M -graded ideal of such divisor D_ρ is

$$I(D_\rho) = \bigoplus_{m \in (\sigma^\vee \setminus \rho^\perp) \cap M} A_m \chi^m, \text{ where } A_m = H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])).$$

A \mathbb{T} -divisor D is horizontal if and only if the induced \mathbb{T} -action on D is of complexity one.

- *Vertical* divisors denoted by $D_{(z,v)}$, where $z \in C$ and $v \in \Delta_z(0)$ is a vertex of Δ_z . The associated M -graded ideal is

$$I(D_{(z,v)}) = \bigoplus_{m \in \sigma^\vee \cap M} (A_m \cap \{f \in \mathbb{K}(C) \mid \operatorname{ord}_z f > -v(m)\}) \chi^m.$$

A \mathbb{T} -divisor D is vertical if and only if the induced \mathbb{T} -action on D is of complexity zero.

Moreover, if $f\chi^m \in A$ is homogeneous of degree m , then the corresponding principal divisor is given by the formula

$$\operatorname{div}(f\chi^m) = \operatorname{div}(f\chi^m)_{\text{hor}} + \operatorname{div}(f\chi^m)_{\text{ver}}$$

where

$$\begin{aligned} \operatorname{div}(f\chi^m)_{\text{hor}} &= \sum_{\rho \in \sigma(1)^*} \rho(m) \cdot D_\rho, \\ \operatorname{div}(f\chi^m)_{\text{ver}} &= \sum_{z \in C} \sum_{v \in \Delta_z(0)} \mu(v) (v(m) + \operatorname{ord}_z f) \cdot D_{(z,v)}, \end{aligned}$$

see [PS11, Proposition 3.14]. The divisor $\operatorname{div}(f\chi^m)_{\text{hor}}$ (resp. $\operatorname{div}(f\chi^m)_{\text{ver}}$) will be called the *horizontal part* (resp. *vertical part*) of $\operatorname{div}(f\chi^m)$.

2.2. \mathbb{G}_a -actions on affine \mathbb{T} -varieties of complexity one.

In this subsection, we review results in [Lie10a] concerning the classification of normalized additive group actions on affine \mathbb{T} -varieties of complexity one.

2.6. Let Z be a variety. Let

$$K \times Z \rightarrow Z, (g, x) \mapsto g \cdot x$$

be an action of an algebraic group K on Z . An additive group action

$$\gamma : \mathbb{G}_a \times Z \rightarrow Z, (\lambda, x) \mapsto \lambda * x$$

is called *normalized* by K if there exists a character $\chi : K \rightarrow \mathbb{G}_m$ such that

$$g \cdot (\lambda * (g^{-1} \cdot x)) = (\lambda \chi^{-1}(g)) * x$$

for all $x \in Z$, $g \in K$, and $\lambda \in \mathbb{G}_a$. The character χ will be called a *degree* of γ . Note that if γ is non-trivial, then χ is uniquely determined by γ .

Consider the algebraic group $\mathbb{G}_a \rtimes_\chi K$, where the underlying variety is $\mathbb{G}_a \times K$ and the group law is given by

$$(\lambda_1, g_1) \cdot (\lambda_2, g_2) = (\lambda_1 + \chi^{-1}(g_1)\lambda_2, g_1 \cdot g_2)$$

for $\lambda_i \in \mathbb{G}_a$ and $g_i \in K$; this algebraic group is the semi-direct product of \mathbb{G}_a and K associated to the character χ . Moreover, a K -action on Z and an additive group action normalized by K of degree χ naturally generate the action of $\mathbb{G}_a \rtimes_\chi K$ on Z via the formula

$$(\lambda, g) \cdot x = \lambda * (g \cdot x),$$

where $(\lambda, g) \in \mathbb{G}_a \rtimes_\chi K$ and $x \in X$. If K is the torus \mathbb{T} and $e \in M$ is lattice vector, then we denote $\mathbb{G}_a \rtimes_e \mathbb{T} = \mathbb{G}_a \rtimes_{\chi^e} \mathbb{T}$.

The following definition is introduced in [FZ05] to describe normal quasi-homogeneous affine surfaces and extended in [Lie10a, Section 1.2] to varieties with a \mathbb{T} -action.

Definition 2.7. Let Z be a variety endowed with a \mathbb{T} -action. An additive group action on Z normalized by \mathbb{T} is called *vertical* if a general \mathbb{G}_a -orbit of Z is contained in the closure of a \mathbb{T} -orbit. Otherwise, the action is called *horizontal*.

2.8. For an affine variety Z , there is a well-known one-to-one correspondence between the \mathbb{G}_a -actions on Z and the locally nilpotent derivations on $\mathbb{K}[Z]$ (called LNDs for brevity). Indeed, let $\mathbb{K}[\mathbb{G}_a] = \mathbb{K}[\lambda]$ for a variable λ over \mathbb{K} . Then, given an LND ∂ on $\mathbb{K}[Z]$, the morphism

$$\mathbb{K}[Z] \rightarrow \mathbb{K}[Z] \otimes_{\mathbb{K}} \mathbb{K}[\lambda], \quad f \mapsto \sum_{i \geq 0} \frac{\partial^i(f)}{i!} \otimes \lambda^i,$$

defines a \mathbb{G}_a -action γ_∂ on Z , and vice-versa. See [Fre06] for details concerning the theory of LNDs. Now assume that Z is endowed with a \mathbb{T} -action. It is also known that \mathbb{G}_a -actions on Z normalized by \mathbb{T} of degree χ^e correspond to homogeneous LNDs of degree e with respect to the M -grading on $\mathbb{K}[Z]$ (see the proof of [FZ05, Lemma 2.2]).

For the next paragraphs, we keep the same notation as above for the σ -polyhedral divisor \mathfrak{D} . We let $X = \text{Spec } A[C, \mathfrak{D}]$ be the associated affine \mathbb{T} -variety of complexity one.

2.9. Let ∂ be a homogeneous LND on $\mathbb{K}[X]$. We have the following characterization.

- γ_∂ is vertical if and only if $\mathbb{K}(C) \subset \mathbb{K}(X)^\partial$, where $\mathbb{K}(X)^\partial \subset \mathbb{K}(X)$ is the field of invariants.
- γ_∂ is horizontal if and only if $\mathbb{K}(C) \cap \mathbb{K}(X)^\partial = \mathbb{K}$. In other words, the algebraic group $\mathbb{G}_a \rtimes_e \mathbb{T}$ acts on X with an open orbit.

We call the corresponding LNDs *vertical* and *horizontal* respectively. The set of non-zero vertical (resp. horizontal) LNDs on $\mathbb{K}[X]$ is denoted by $\text{LND}_{\text{ver}}(X)$ (resp. $\text{LND}_{\text{hor}}(X)$).

The following paragraph presents the classical description of vertical homogeneous LNDs in complexity one. See [Lie10b] for the general case.

2.10. Recall from [Dem70, Section 4.5, p. 571] that a vector $e \in M$ is a *Demazure root* of the cone σ with distinguished ray $\rho_e \in \sigma(1)$ if $\rho_e(e) = -1$ and $\rho(e) \geq 0$ for all $\rho \in \sigma(1) \setminus \{\rho_e\}$. We denote by $\text{Rt}(\sigma)$ the set of all roots of σ . If $e \in \text{Rt}(\sigma)$, then we let

$$\Phi_e^* = H^0(C, \mathcal{O}_C([\mathfrak{D}(e)])) \setminus \{0\}.$$

Furthermore, $\Phi_e^* \neq \emptyset$ if and only if $\rho_e \in \sigma(1)^*$, see [Lie10a, Corollary 3.13]. Assume that Φ_e^* has an element φ . We define a homogeneous LND $\partial_{e,\varphi}$ of vertical type on the algebra $A = A[C, \mathfrak{D}]$ by the formula

$$\partial_{e,\varphi}(f\chi^m) = \rho_e(m)\varphi f\chi^{m+e},$$

where $f\chi^m \in A$ is an homogeneous element of degree m , see [Lie10a, Lemma 3.6]. More precisely, let

$$\text{Rt}^*(\sigma) = \{e \in \text{Rt}(\sigma) \mid \rho_e \in \sigma(1)^*\}.$$

Considering the set

$$\Phi = \{(e, \varphi) \mid e \in \text{Rt}^*(\sigma), \varphi \in \Phi_e^*\},$$

the map $\Phi \rightarrow \text{LND}_{\text{ver}}(A)$, $(e, \varphi) \mapsto \partial_{e,\varphi}$ is a bijection [Lie10a, Theorem 3.8].

In the sequel, we consider the case of horizontal homogeneous LNDs in complexity one. Note that if $X = \text{Spec } A[C, \mathfrak{D}]$ admits a normalized \mathbb{G}_a -action of horizontal type, then either $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$ (see [Lie10a, 3.16]). The next paragraph introduces combinatorial objects that describe homogeneous LNDs of horizontal type on $A = A[C, \mathfrak{D}]$ (see [Lie10a, Section 3.2], [AL12, Section 1.4], [Lan13, Section 5.6], and [LL]). Note that our terminology will be slightly different.

2.11. Let \mathfrak{D} be a proper σ -polyhedral divisor over \mathbb{A}^1 . An (*affine*) *AL-coloring*² on the polyhedral divisor \mathfrak{D} is a triple $\mathfrak{D}_\bullet = (\mathfrak{D}, \{v_z^\bullet \mid z \in \mathbb{A}^1\}, z_0)$, where $z_0 \in \mathbb{A}^1$, satisfying the following conditions:

- each v_z^\bullet is a vertex of Δ_z ;
- $v_{\deg}^\bullet := \sum_{z \in \mathbb{A}^1} v_z^\bullet$ is a vertex of the polyhedron $\deg \mathfrak{D} = \sum_{z \in \mathbb{A}^1} \Delta_z$;
- $v_z^\bullet \in N$ for any $z \in \mathbb{A}^1 \setminus \{z_0\}$.

The elements v_z^\bullet for z in the support of \mathfrak{D} are called the *AL-colors* of \mathfrak{D}_\bullet . The point z_0 is called the *marked point* of \mathfrak{D}_\bullet . The *support* of \mathfrak{D}_\bullet is defined as the union of the support of \mathfrak{D} and of the marked point z_0 . The polyhedron

$$\omega_\bullet = \text{Cone}(\deg \mathfrak{D} - v_{\deg}^\bullet)$$

is called the *associated cone* of \mathfrak{D}_\bullet .

Now let \mathfrak{D} be a proper σ -polyhedral divisor over \mathbb{P}^1 . A (*projective*) *AL-coloring* on the polyhedral divisor \mathfrak{D} is a quadruple $\mathfrak{D}_\bullet = (\mathfrak{D}, \{v_z^\bullet \mid z \in \mathbb{A}^1\}, z_0, z_\infty)$, where $z_\infty \in \mathbb{P}^1$, $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{z_\infty\}$, and the triple $\mathfrak{D}_\bullet^{\text{aff}} = (\mathfrak{D}|_{\mathbb{A}^1}, \{v_z^\bullet \mid z \in \mathbb{A}^1\}, z_0)$ is an affine AL-coloring. We define the AL-colors, the marked point, the associated cone, and the support of \mathfrak{D}_\bullet as those of $\mathfrak{D}_\bullet^{\text{aff}}$. We also call z_∞ the *point at the infinity* of \mathfrak{D}_\bullet .

²named by I. Arzhantsev and A. Liendo. The sign \bullet will be substituted by $-$ and $+$ later on, when we consider different AL-colorings of a polyhedral divisor.

Remark 2.12. The definition of ω_\bullet is motivated by the following relation:

$$(1) \quad \mathfrak{D}(m)|_{\mathbb{A}^1} = \sum_{z \in \mathbb{A}^1} v_z^\bullet(m) \cdot z \text{ if and only if } m \in \omega_\bullet^\vee.$$

2.13. Since we will consider several colorings on a same polyhedral divisor, it is convenient to represent them by a weighted graph. Let $\{z_0, \dots, z_r\} \subset \mathbb{A}^1$ be the support of \mathfrak{D}_\bullet , where z_0 is the marked point. A representation of \mathfrak{D}_\bullet will be drawn as

$$\begin{array}{ccccccc} & v_{z_0}^\bullet & v_{z_1}^\bullet & & v_{z_r}^\bullet & & \\ & \circ & \circ & \cdots & \circ & & \\ \cdots & \text{---} & \text{---} & \cdots & \text{---} & \cdots & \\ & z_0 & z_1 & & z_r & & \end{array}$$

where $v_{z_0}^\bullet, \dots, v_{z_r}^\bullet$ are the corresponding AL-colors of \mathfrak{D}_\bullet .

Let us introduce other objects attached to a coloring of \mathfrak{D} . In the following definition, we keep the same notation as above for the AL-coloring \mathfrak{D}_\bullet .

Definition 2.14. Let $\tilde{\omega}_\bullet \subset (N \oplus \mathbb{Z})_\mathbb{Q}$ be the cone generated by $(\omega_\bullet, 0)$ and $(v_{z_0}^\bullet, 1)$ if $C = \mathbb{A}^1$, and by $(\omega_\bullet, 0)$, $(v_{z_0}^\bullet, 1)$, and $(\Delta_{z_\infty} + v_{\deg}^\bullet - v_{z_0}^\bullet + \omega_\bullet, -1)$ if $C = \mathbb{P}^1$. The polyhedral cone $\tilde{\omega}_\bullet$ will be called the *domain* of \mathfrak{D}_\bullet . Let $e \in M$. A pair $(\mathfrak{D}_\bullet, e)$ is said to be *coherent* if the following conditions are satisfied.

- Let $s = -\frac{1}{d(e)} - v_{z_0}^\bullet(e)$, where $d(e) = \inf\{d \in \mathbb{Z}_{>0} \mid dv_{z_0}^\bullet(e) \in \mathbb{Z}\}$. The vector (e, s) is a Demazure root of $\tilde{\omega}_\bullet$;
- $v(e) \geq 1 + v_z^\bullet(e)$ for any $z \in \mathbb{A}^1 \setminus \{z_0\}$ and for any $v \in \Delta_z(0) \setminus \{v_z^\bullet\}$;
- $v(e) \geq -s$ for any $v \in \Delta_{z_0} \setminus \{v_{z_0}^\bullet\}$;
- If $C = \mathbb{P}^1$, then $v(e) \geq -\frac{1}{d(e)} - v_{\deg}^\bullet(e)$ for any $v \in \Delta_{z_\infty}(0)$.

The next theorem provides a description of normalized \mathbb{G}_a -actions of horizontal type on the \mathbb{T} -variety $X = \text{Spec } A[C, \mathfrak{D}]$ in terms of coherent pairs [AL12, Theorem 1.10]. We fix a variable t over \mathbb{K} such that $z_0 = \{t = 0\}$ and $z_\infty = \{t = \infty\}$ (if exists), in particular $\mathbb{K}[\mathbb{A}^1] = \mathbb{K}[t]$.

Theorem 2.15. *Let \mathfrak{D} be a proper σ -polyhedral divisor over $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$ and let $A = A[C, \mathfrak{D}]$.*

i) Consider a triplet $(\lambda, \mathfrak{D}_\bullet, e)$, where $\lambda \in \mathbb{K}^$, \mathfrak{D}_\bullet is an AL-coloring of \mathfrak{D} , and $(\mathfrak{D}_\bullet, e)$ is a coherent pair. Assume without loss of generality that $v_z^\bullet = 0$ for all $z \in \mathbb{A}^1 \setminus \{0\}$. Each such triplet $(\lambda, \mathfrak{D}_\bullet, e)$ defines a horizontal LND ∂ on A via the formula*

$$\partial(t^r \chi^m) = \lambda \cdot d(e) \cdot (v_0^\bullet(m) + r) \cdot t^{r+s} \chi^{m+e} \text{ for all } (m, r) \in M \oplus \mathbb{Z},$$

where $s = -\frac{1}{d(e)} - v_0^\bullet(e)$. The kernel of the LND ∂ corresponding to the triplet $(\lambda, \mathfrak{D}_\bullet, e)$ is

$$\ker \partial = \bigoplus_{m \in \omega_\bullet^\vee \cap L} \mathbb{K} \varphi_m \chi^m,$$

where $L = \{m \in M \mid v_0^\bullet(m) \in \mathbb{Z}\}$, $\varphi_m \in \mathbb{K}(C)^$ such that $(\text{div}(\varphi_m) + \mathfrak{D}(m))|_{\mathbb{A}^1} = 0$, and ω_\bullet is the associated cone of \mathfrak{D}_\bullet . Moreover, ω_\bullet^\vee is a maximal cone of $\Lambda(\mathfrak{D}|_{\mathbb{A}^1})$ (see 2.2).*

ii) All horizontal LNDs on A are obtained in this way.

Remark 2.16. More generally, if z_0 is arbitrary, $z_\infty = \infty$, and \mathfrak{D}_\bullet be represented by

$$\begin{array}{ccccccc} & v_{z_0}^\bullet & v_{z_1}^\bullet & & v_{z_k}^\bullet & & \\ & \circ & \circ & \cdots & \circ & & \\ \cdots & \text{---} & \text{---} & \cdots & \text{---} & \cdots & \\ & z_0 & z_1 & & z_k & & \end{array}$$

then the LND ∂_\bullet is given by the formula

$$\partial_\bullet((t - z_0)^r \xi_m \chi^m) = d(e)(v_{z_0}^\bullet(m) + r)(t - z_0)^{r-s} \xi_{m+e} \chi^{m+e},$$

where

$$\xi_m = \prod_{i=1}^k (t - z_i)^{-v_{z_i}^\bullet(m)} \text{ and } s = -\frac{1}{d(e)} - v_{z_0}^\bullet(e).$$

2.3. Horizontal SL_2 -actions.

In this subsection, we recall generalities on compatible SL_2 -actions (see [AL12, Section 2]) of horizontal type for affine \mathbb{T} -varieties of complexity one.

2.17. We make the convention that SL_2 is the algebraic group of 2×2 matrices over \mathbb{K} with determinant one. For every $e \in M$ we define the reductive group $G_e = \mathrm{SL}_2 \rtimes_e \mathbb{T}$ as follows. The underlying variety is $\mathrm{SL}_2 \times \mathbb{T}$ and the multiplication law is given by the relation

$$(A, t_1) \cdot (B, t_2) = (A \cdot \varepsilon_e(t_1)(B), t_1 \cdot t_2),$$

where $(A, t_1), (B, t_2) \in G_e$ and the function ε_e satisfies

$$\varepsilon_e(t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \chi^{-e}(t)b \\ \chi^e(t)c & d \end{pmatrix}$$

for all $t \in \mathbb{T}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$. The group G_e is the semi-direct product of SL_2 and \mathbb{T} via the map ε_e .

2.18. In the sequel, we let $T \subset \mathrm{SL}_2$ be the maximal torus which consists of diagonal matrices. We also consider the root subgroups

$$U_- = \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{K} \right\} \text{ and } U_+ = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{K} \right\}$$

with respect to T . We make also the convention that $\mathrm{SL}_2 \subset G_e$ via the natural isomorphism $\mathrm{SL}_2 \cong \mathrm{SL}_2 \times \{1\}$. We denote by $B_- \subset G_e$ (resp. $B_+ \subset G_e$) the Borel subgroup generated by U_- (resp. U_+) and $T \times \mathbb{T}$.

In the following lemma we describe the subgroups of G_e .

Lemma 2.19. I. *The group G_e possesses a non-trivial center*

$$(2) \quad Z(G_e) = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, t \right) \mid \chi^e(t) = a^2, a, t \in \mathbb{K}^* \right\}.$$

II. *Any closed subgroup of G_e up to conjugation is of one of the following types:*

- (i) “center extension”, i.e. a subgroup $H \subset G_e$ such that the neutral component H° is contained in the center $Z(G)$.
- (ii) quasitorus $Q \subset T \times \mathbb{T}$ such that $Q^\circ \not\subset Z(G_e)$;
- (iii) “normalizing subgroup” $N(Q) = Q \cup \tau Q$ such that $Q^\circ \not\subset Z(G_e)$, where $\tau = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s \right) \in G_e$ satisfies the condition $\tau^2 \in Q$;
- (iv) semidirect product $U_+ \rtimes Q$, where the quasitorus Q is a subgroup of $T \times \mathbb{T}$;
- (v) semidirect product $\mathrm{SL}_2 \rtimes Q'$, where the quasitorus Q' is a subgroup of \mathbb{T} .

III. *The spherical subgroups of G_e are those with a non-central neutral component, i.e. of types (ii)–(v).*

Proof. Let (A, t) be an arbitrary element in G_e . Conjugating by $(B, t^{-1}) \in G_e$, we obtain

$$(3) \quad (B \cdot \varepsilon_e(t^{-1})(A \cdot \varepsilon_e(t)(B^{-1})), t) = (B \cdot \varepsilon_e(t^{-1})(A) \cdot B^{-1}, t).$$

It is easy to see that up to conjugation we may assume A to be upper (or lower) triangular for any (A, t) , and diagonal for a semisimple element $(A, t) \in G_e$. In particular, if (A, t) is central, then A is diagonal.

Let us study the center $Z(G)$. Consider the conjugation of a diagonal element $z = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, b\right) \in G$ by an arbitrary $g = \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) \in G$:

$$(4) \quad g \cdot z \cdot g^{-1} = \left(\begin{pmatrix} axw - a^{-1}yz\chi^e(b), & (a^{-1} - a\chi^e(b^{-1}))xy \\ (a - a^{-1}\chi^e(b))wz, & a^{-1}xw - ayz\chi^e(b^{-1}) \end{pmatrix}, b\right).$$

So, $z \in Z(G)$ if and only if $a^2 = \chi^e(b)$. So, the first statement of the lemma is proved.

Let $H \subset G_e$ be a closed subgroup. Assume that all its elements are semisimple. Then the connected component is diagonalizable, so up to conjugation $H^\circ \subset T \times \mathbb{T}$.

If $H^\circ \subset Z(G)$, then our group is of type (i), i.e. a “center extension”. Otherwise, there exists $h = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, b\right) \in H^\circ$ such that $h \notin Z(G_e)$. Since H° is normal in H , for any $g = \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) \in H$ the conjugation $g \cdot h \cdot g^{-1}$ lies in H° and in particular is diagonal. By (4), $xy = wz = 0$. This is possible if g is either of form $\left(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, *\right)$ or of form $\left(\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, *\right)$. Since $Q = H \cap (T \times \mathbb{T})$ is a quasitorus, either $H = Q$ or $H = Q \cup \tau Q$ for some $\tau = \left(\begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}, b\right)$. In the latter case, $\tau^2 \in Q$. In addition, if we conjugate H by a diagonal element, then Q is preserved and τ can be reduced to form $\tau = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b\right)$.

Now assume there is $h \in H$ that is not semisimple. By (3), we may assume up to conjugation that h is upper-triangular. Consider the Jordan decomposition $h = h_s h_u$ into the semisimple part h_s and the unipotent part h_u . Since H is closed, $h_u \in H$ and H contains U_+ .

Let H be contained in the Borel subgroup $\{(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, *)\}$. Then any element of H is decomposed into an element of U and an element of $T \times \mathbb{T}$. So, $H = U \rtimes Q$, where $Q \subset T \times \mathbb{T}$ is a quasitorus.

Assume now that there exists $h \in H$ that is not contained in the Borel subgroup. Then by direct computation we obtain that the subset

$$U_+ \cdot h \cdot U_+ \cdot h^{-1} \cdot U_+ \subset \mathrm{SL}_2$$

is three-dimensional and dense in SL_2 . Thus, $H = \mathrm{SL}_2 \rtimes Q$, $Q \subset \mathbb{T}$.

Finally, let us describe spherical subgroups. A subgroup $H \subset G_e$ is spherical if and only if the neutral component H° is spherical. The connected subgroup H° is spherical if and only if $B \cdot H^\circ$ is dense in G_e for some Borel subgroup B . Since $B \subset G_e$ is of codimension one, the latter condition is equivalent to $H^\circ \not\subset B$. So, a subgroup $H \subset G_e$ is spherical if and only if $H^\circ \not\subset Z(G_e)$. \square

Remark 2.20. In fact, normal closed subgroups of G_e are exhausted by:

- (1) central quasitori $Q \subset Z(G_e)$;
- (2) semidirect products $\mathrm{SL}_2 \rtimes Q'$, where $Q' \subset \mathbb{T}$.

2.21. Let us describe a G_e -action on an affine variety Z . This is equivalent to having (see [CD03, Theorem 3])

- an M -grading on $\mathbb{K}[Z]$, and

- three derivations $\partial_-, \delta, \partial_+$ on $\mathbb{K}[Z]$ satisfying

$$[\delta, \partial_{\pm}] = \pm 2\partial_{\pm} \text{ and}$$

$$[\partial_+, \partial_-] = \partial_+ \circ \partial_- - \partial_- \circ \partial_+ = \delta,$$

where ∂_{\pm} is a homogeneous LND of degree $\pm e$ and δ is a semi-simple derivation inducing a downgrading of the M -grading.

Note that ∂_-, ∂_+ , and δ are given by the induced actions of U_-, U_+ , and T respectively, whereas the M -grading corresponds to the induced \mathbb{T} -action. Assume that the \mathbb{T} -action on Z is of complexity at least one. Then the variety Z is G_e -spherical if and only if ∂_- and ∂_+ are both of horizontal type.

2.22. By [AL12, Corollary 2.3 (ii)], in case $e = 0$ we may consider the bigger torus and fall into $e \neq 0$. So, the classification of affine G_e -spherical varieties falls into the following categories:

- (1) $e \neq 0$ and Z is not toric for the action of a larger torus than \mathbb{T} .
- (2) $e \neq 0$ and Z is toric for the \mathbb{T} -action.

Definition 2.23. An affine G_e -spherical variety Z satisfying (1) (resp. (2)) is called of *type I* (resp. of *type II*).

The following theorem gives a complete description of affine G_e -spherical varieties of type I in terms of polyhedral divisors (see [AL12, Theorem 2.18]). Here t is a local parameter on \mathbb{P}^1 and $0, 1, \infty \in \mathbb{P}^1$ are chosen accordingly.

Theorem 2.24. *Let X be an affine variety with a G_e -action such that the induced \mathbb{T} -action is faithful. Then X is G_e -spherical of type I if and only if $\mathbb{K}[X]$ is G_e -isomorphic to a rational G_e -algebra $A = A[C, \mathfrak{D}]$ enjoying the following properties.*

- (a) $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ -polyhedral divisor over $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$ supported in at most three points $0, 1, \infty$.
- (b) The horizontal LNDs ∂_-, ∂_+ are given by the coherent pairs³ $(\mathfrak{D}_-, -e)$, (\mathfrak{D}_+, e) , where the marked point of \mathfrak{D}_{\pm} is 0 and the point at the infinity of \mathfrak{D}_{\pm} is ∞ .
- (c) $\mathfrak{D}, \mathfrak{D}_-,$ and \mathfrak{D}_+ are defined by the following conditions separated in two cases.
 - *Either (reflexive case) $\Delta_0 = \text{Conv}(0, v_0) + \sigma$, $\Delta_1 = \text{Conv}(0, v_1) + \sigma$, where $v_0, v_1 \in N \setminus \sigma$, $v_0(e) = 1$, $v_1(e) = -1$, $\Delta_{\infty}(0) \subset e^{\perp}$, and⁴ $(\pm e, -\frac{1}{2} \mp \frac{1}{2}) \in \text{Rt}(\tilde{\omega}_{\pm})$. Representations of $\mathfrak{D}_-, \mathfrak{D}_+$ are given respectively by*

$$\begin{array}{ccccccc} & v_0 & 0 & & & 0 & v_1 \\ \cdots & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots \\ & 0 & 1 & & & 0 & 1 \end{array}$$

- *Or (skew case) $\Delta_0 = v_0 + \sigma$, $\Delta_1 = \text{Conv}(0, v_1) + \sigma$, where $2v_0, v_1 \in N \setminus \sigma$, $2v_0(e) = 1$, $v_1(e) = -1$, $\Delta_{\infty}(0) \subset e^{\perp}$, and $(\pm e, -\frac{1}{2} \mp \frac{1}{2}) \in \text{Rt}(\tilde{\omega}_{\pm})$. Representations of $\mathfrak{D}_-, \mathfrak{D}_+$ are given respectively by*

$$\begin{array}{ccccccc} & v_0 & 0 & & & v_0 & v_1 \\ \cdots & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots \\ & 0 & 1 & & & 0 & 1 \end{array}$$

³By the abuse of notation we allow a coherent pair $(\mathfrak{D}_{\bullet}, e)$ to denote a triplet $(1, \mathfrak{D}_{\bullet}, e)$, cf. Theorem 2.15.

⁴The polyhedral cone $\tilde{\omega}_{\pm}$ is the domain of \mathfrak{D}_{\pm} (see 2.14).

Remark 2.25. The LND ∂_{\pm} corresponding to the coherent pair $(\mathfrak{D}_{\pm}, \pm e)$ in Theorem 2.24 is defined via the formulae (see the comment before [AL12, Lemma 2.16] and the proof of [AL12, Lemma 2.17])

either (reflexive case)

$$\begin{aligned}\partial_{\pm}(t^r \xi_m^{\pm} \chi^m) &= \left[\left(\frac{1}{2} \mp \frac{1}{2} \right) v_0(m) + r \right] \cdot t^{r - \frac{1}{2} \mp \frac{1}{2}} \xi_{m \pm e}^{\pm} \chi^{m \pm e} \text{ for all } (m, r) \in M \oplus \mathbb{Z}, \\ \delta(t^r \chi^m) &= (v_0 - v_1)(m) \cdot t^r \chi^m \text{ for all } (m, r) \in M \oplus \mathbb{Z},\end{aligned}$$

where $\xi_m^{\pm} = (t - 1)^{-(\frac{1}{2} \pm \frac{1}{2})v_1(m)}$;

or (skew case)

$$\begin{aligned}\partial_{\pm}(t^r \xi_m^{\pm} \chi^m) &= 2(v_0(m) + r) \cdot t^{r - \frac{1}{2} \mp \frac{1}{2}} \xi_{m \pm e}^{\pm} \chi^{m \pm e} \text{ for all } (m, r) \in M \oplus \mathbb{Z}, \\ \delta(t^r \chi^m) &= -2v_1(m) \cdot t^r \chi^m \text{ for all } (m, r) \in M \oplus \mathbb{Z},\end{aligned}$$

where ξ_m^{\pm} is again $(t - 1)^{-(\frac{1}{2} \pm \frac{1}{2})v_1(m)}$.

Remark 2.26. The formulas for ∂_{\pm} can also be expressed for an arbitrary $Q \in \mathbb{K}(t)$. In reflexive case they are:

$$\begin{aligned}\partial_{-}(Q \chi^m) &= (v_0(m)Q + tQ') \chi^{m-e}, \\ \partial_{+}(Q \chi^m) &= (v_1(m)Q + (t-1)Q') \chi^{m+e},\end{aligned}$$

where $Q' = \frac{dQ}{dt}$. In skew case they are:

$$\begin{aligned}\partial_{-}(Q \chi^m) &= 2(v_0(m)Q + tQ') \chi^{m-e}, \\ \partial_{+}(Q \chi^m) &= 2 \left(\left(v_0(m) \left(1 - \frac{1}{t} \right) + v_1(m) \right) Q + (t-1)Q' \right) \chi^{m+e}.\end{aligned}$$

3. COLORED CONES VERSUS COLORED POLYHEDRAL DIVISORS

Let G_e be the previously introduced reductive group $\mathrm{SL}_2 \rtimes_e \mathbb{T}$. The purpose of this section is to express the approach of Arzhantsev–Liendo in terms of the Luna–Vust theory applied to affine G_e -spherical varieties. Sections 3.1 and 3.2 are devoted to the type I and Section 3.3 to the type II.

3.1. Classification of the type I.

Let X be an affine G_e -spherical variety of type I. Suppose that $\mathbb{K}[X] = A[C, \mathfrak{D}]$ admits a presentation as in 2.24 (either skew or reflexive case). We identify the open G_e -orbit of X with a homogeneous space G_e/H so that X is its embedding.

Remark 3.1. Since X is not horospherical [AL12, Proposition 3.8], Lemma 2.19 implies that H is reductive. By the Matsushima criterion, G_e/H is affine.

3.2. Denote by

$$\begin{array}{ccccccc} & v_0^- & v_1^- & & & v_0^+ & v_1^+ \\ \cdots & \text{---} \circ & \text{---} \circ & \cdots & \cdots & \text{---} \circ & \text{---} \circ & \cdots \\ & 0 & 1 & & & 0 & 1 & \end{array}$$

the representations of \mathfrak{D}_{-} and \mathfrak{D}_{+} respectively. For instance, in the reflexive case, $v_0^- = v_0$, $v_1^- = 0$, $v_0^+ = 0$, and $v_1^+ = v_1$. For every $m \in M$ we let

$$\varphi_m^{\pm} = (t - 1)^{-v_1^{\pm}(m)} t^{-\lfloor v_0^{\pm}(m) \rfloor}.$$

Then we have

$$(5) \quad \ker \partial_{\pm} = \bigoplus_{m \in \omega_{\pm}^{\vee} \cap L} \mathbb{K} \varphi_m^{\pm} \chi^m,$$

where ω_{\pm} is the associated cone of \mathfrak{D}_{\pm} and

$$L = \{m \in M \mid v_0^+(m) \in \mathbb{Z}\} = \{m \in M \mid v_0^-(m) \in \mathbb{Z}\}.$$

In the following lemma we use the notation from 2.18.

Lemma 3.3. *Let $A_M^{\pm} = \bigoplus_{m \in M} \mathbb{K}[t] \cdot \varphi_m^{\pm} \chi^m$. Then the following assertions hold.*

- (i) *The lattice of B_{\pm} -weights of the B_{\pm} -algebra $\mathbb{K}(X)$ is L .*
- (ii) *The variety $X_M^{\pm} = \text{Spec } A_M^{\pm}$ is a principal open subset of X which coincides with the open B_{\pm} -orbit.*

More precisely, the supports of all divisors $\text{div}(\varphi_m^{\pm} \chi^m)$ for $m \in \text{rel.int.}(\omega_{\pm}^{\vee}) \cap L$ are equal and form the complement $X \setminus X_M^{\pm}$.

Proof. (i) The first assertion results from the fact that B_{\pm} -eigenvectors in $\mathbb{K}(X)$ are exactly quotients of non-zero homogeneous elements of $\ker \partial_{\pm}$.

(ii) By Remark 2.12, we can add a principal polyhedral divisor to \mathfrak{D} so that for any $m \in \text{rel.int.}(\omega_{\pm}^{\vee}) \cap L$ the divisor $\mathfrak{D}(m)|_{\mathbb{A}^1}$ is supported in at most the point 0 and that $\varphi_m^{\pm} = t^{-[v_0^{\pm}(m)]}$. Let us fix $m \in \text{rel.int.}(\omega_{\pm}^{\vee}) \cap L$ and denote $\alpha_{\pm} = \varphi_m^{\pm} \chi^m$. By [AHS08, Proposition 3.3], the localization $\mathbb{K}[X]_{\alpha_{\pm}}$ is equal to A_M^{\pm} . Since $\alpha_{\pm} \in \ker \partial_{\pm}$, the derivation ∂_{\pm} extends to an LND on A_M^{\pm} written by the same letter ∂_{\pm} . Hence, X_M^{\pm} is a B_{\pm} -stable principal open subset of X .

Let us deduce that X_M^{\pm} is the open B_{\pm} -orbit. Denote by E_M^{\pm} the normalization of the algebra $A_M^{\pm}[\sqrt[d]{\varphi_{\pm de}^{\pm} \chi^{\pm e}}]$, where $d = d(e)$. Then the inclusion $A_M^{\pm} \subset E_M^{\pm}$ yields a finite surjective map

$$\pi : Z_M^{\pm} = \text{Spec } E_M^{\pm} \rightarrow X_M^{\pm}.$$

By [Lan13, Corollary 5.3.6], the derivation ∂_{\pm} admits a unique extension to E_M^{\pm} , hence there is a natural B_{\pm} -action on Z_M^{\pm} and π is B_{\pm} -equivariant. Denote $\zeta = \sqrt[d]{t}$. The map

$$\gamma : E_M^{\pm} \rightarrow \mathbb{K}[M][\zeta] = \bigoplus_{m \in M} \mathbb{K}[\zeta] \chi^m, \quad \zeta^l \chi^m \mapsto \zeta^{l+dv_0^{\pm}(m)} \chi^m$$

is an isomorphism of M -graded algebras. Consider the extension of ∂_{\pm} to E_M^{\pm} denoted by $\tilde{\partial}_{\pm}$ and introduce $\tilde{\partial}'_{\pm} = \gamma \circ \tilde{\partial}_{\pm} \circ \gamma^{-1}$, which is a homogeneous LND on $\mathbb{K}[M][\zeta]$. Then the induced B_{\pm} -action on $\text{Spec } \mathbb{K}[M][\zeta] \cong \mathbb{T} \times \mathbb{A}^1$ is transitive, and so is the B_{\pm} -action on Z_M^{\pm} . Since π is B_{\pm} -equivariant and surjective, the B_{\pm} -action on X_M^{\pm} is also transitive. \square

Now we give an explicit description of the B_{\pm} -divisors on X and compute their images in the vector space $V_{\mathbb{Q}}$ (see 1.4, 1.5), where $V = \text{Hom}(L, \mathbb{Z})$ is the valuation lattice. Note that $V_{\mathbb{Q}}$ is identified with $N_{\mathbb{Q}}$.

Proposition 3.4. *Denote by ϱ_{\pm} the map that sends a discrete valuation ν of $\mathbb{K}(X)$ into the linear form $\varrho_{\pm}(\nu) \in V_{\mathbb{Q}}$ defined as follows. For any B_{\pm} -eigenvector $\alpha \in \mathbb{K}(X)$ of weight $\chi_{\alpha} \in L$ we let*

$$\varrho_{\pm}(\nu)(\chi_{\alpha}) = \nu(\alpha).$$

To simplify the notation, we also let $\varrho(D) = \varrho(v_D)$ for a prime divisor $D \subset X$. Then the colors of X are given in the following table in accordance with notation 2.5.

	Reflexive case	Skew case
B_- -action	$\varrho_-(D_{(0,0)}) = -v_0, \varrho_-(D_{(1,v_1)}) = v_1$	$\varrho_-(D_{(1,v_1)}) = v_1$
B_+ -action	$\varrho_+(D_{(0,v_0)}) = v_0, \varrho_+(D_{(1,0)}) = -v_1$	$\varrho_+(D_{(1,0)}) = -v_1$

Moreover, the G_e -divisors on X are exhausted by D_ρ for $\rho \in \sigma(1)^\star$ and $D_{(\infty,v)}$ for $v \in \Delta_\infty(0)$. We have $\varrho_\pm(D_\rho) = \rho$ for all $\rho \in \sigma(1)^\star$. The images of the vertical G_e -divisors are given in the following table.

	Reflexive case	Skew case
B_- -action	$\varrho_-(D_{(\infty,v)}) = \mu(v)(v_0 + v)$	$\varrho_-(D_{(\infty,v)}) = \mu(v)(v_0 + v)$
B_+ -action	$\varrho_+(D_{(\infty,v)}) = \mu(v)(v_1 + v)$	$\varrho_+(D_{(\infty,v)}) = \mu(v)(v_0 + v_1 + v)$

Proof. We will illustrate our calculation method on the B_+ -action in the reflexive case. It can be easily adapted to the remaining cases that are left to the reader.

A representation of \mathfrak{D}_+ in the reflexive case is given by

$$\begin{array}{c} \cdots \quad 0 \quad v_1 \quad \cdots \\ \hline \quad \quad \circ \quad \circ \quad \\ \quad \quad 0 \quad 1 \end{array}$$

Let $m \in \text{rel.int.}(\omega_+^\vee) \cap L$. We have $\varphi_m^+ = (t-1)^{-v_1(m)}$. By Lemma 3.3, the B_+ -divisors of X are exactly the irreducible components of the support of the principal divisor

$$\text{div}(\varphi_m^+ \chi^m) = \sum_{\rho \in \sigma(1)^\star} \rho(m) \cdot D_\rho + \sum_{z \in \{0,1,\infty\}} \sum_{v \in \Delta_z(0)} \mu(v)(v(m) + \text{ord}_z \varphi_m^+) \cdot D_{(z,v)}.$$

Since $m \in \text{rel.int.}(\sigma^\vee)$, the integer $\rho(m)$ is non-zero for every $\rho \in \sigma(1)^\star$. Hence all the horizontal divisors on X are B_+ -stable.

An easy computation shows that the vertical part of $\text{div}(\varphi_m^+ \chi^m)$ is

$$\text{div}(\varphi_m^+ \chi^m)_{\text{vert}} = \Sigma_0 + \Sigma_1 + \Sigma_\infty,$$

where

$$\begin{aligned} \Sigma_0 &= v_0(m) \cdot D_{(0,v_0)}, \\ \Sigma_1 &= -v_1(m) \cdot D_{(1,0)}, \\ \Sigma_\infty &= \sum_{v \in \Delta_\infty(0)} \mu(v)(v + v_1)(m) \cdot D_{(\infty,v)}. \end{aligned}$$

We may choose m such that $v_0(m)$ and $v_1(m)$ are nonzero. By Lemma 3.3, this holds for every $m \in \text{rel.int.}(\omega_+^\vee) \cap L$. Hence the divisors Σ_0 and Σ_1 are nonzero, and so $D_{(0,v_0)}, D_{(1,0)}$ are B_+ -stable. Furthermore, since \mathfrak{D} is proper,

$$\text{ord}_\infty \varphi_m^+ + \min_{v \in \Delta_\infty(0)} v(m) > 0.$$

Thus, for all $v \in \Delta_\infty(0)$, $(v_1 + v)(m) > 0$. This shows that the vertical divisors of the form $D_{(\infty,v)}$ are B_+ -stable.

Making the same computation for the divisor $\text{div}(\varphi_m^- \chi^m)$ where $m \in \text{rel.int.}(\omega_-^\vee) \cap L$, we obtain that the divisors of the form D_ρ and $D_{(\infty,v)}$ are B_- -stable. Hence they form the set of G_e -divisors of X .

The divisors $D_{(0,v_0)}$ and $D_{(1,0)}$ do not appear in the support of $\text{div}(\varphi_m^- \chi^m)$. This implies that $D_{(0,v_0)}, D_{(1,0)}$ are the colors of X for the B_+ -action. Finally, one concludes

by the formula

$$\operatorname{div}(\varphi_m^+ \chi^m) = \sum_{\rho \in \sigma(1)^\star} \varrho_+(D_\rho)(m) \cdot D_\rho + \sum_{z \in \{0,1,\infty\}} \sum_{v \in \Delta_z(0)} \varrho_+(D_{(z,v)})(m) \cdot D_{(z,v)}.$$

□

Remark 3.5. Let us consider the subalgebra $A^h = A[\mathbb{A}^1, \mathfrak{D}_{G_e/H}] \subset \mathbb{K}(t)[M]$ defined by the polyhedral divisor

$$\mathfrak{D}_{G_e/H} = \Delta(G_e/H)_0 \cdot \{0\} + \Delta(G_e/H)_1 \cdot \{1\}$$

with the conditions $\Delta(G_e/H)_0 = \operatorname{Conv}(0, v_0)$, $\Delta(G_e/H)_1 = \operatorname{Conv}(0, v_1)$ in the reflexive case, and $\Delta(G_e/H)_0 = \{v_0\}$, $\Delta(G_e/H)_1 = \operatorname{Conv}(0, v_1)$ otherwise.

Then $\mathbb{K}[X] \subset A^h$, and A^h is G_e -stable as a subalgebra of $\mathbb{K}(X)$. Therefore, $X^h = \operatorname{Spec} A^h$ is an embedding of G_e/H , see 1.2. Since X^h does not include G_e -divisors by Proposition 3.4, and G_e/H is affine by Remark 3.1, $X^h = G_e/H$. So, the polyhedral divisor $\mathfrak{D}_{G_e/H}$ characterizes the open orbit.

The closed G_e -orbit $Y \subset X$ is described as follows.

Lemma 3.6. *The vanishing ideal of Y is*

$$I = \bigoplus_{m \in (\sigma^\vee \setminus \operatorname{lin}(\sigma^\vee)) \cap M} A_m \chi^m, \text{ where } A_m = H^0(C, \mathcal{O}_C(|\mathfrak{D}(m)|)).$$

Proof. Let $R = \mathbb{K}[X]/I$ and $Z = \operatorname{Spec} R$. We will prove that $Z = Y$.

Case $C = \mathbb{A}^1$. Since I is generated by $\bigcup_{\rho \in \sigma(1)} I(D_\rho)$, the ideal I is G_e -stable and R is a rational G_e -algebra. Let us describe R in terms of polyhedral divisors. Consider the projection $\pi : N \rightarrow \tilde{N}$, where $\tilde{N} = N/(\operatorname{lin}(\sigma^\vee)^\perp \cap N)$. The lattice \tilde{N} is the dual of $\operatorname{lin}(\sigma^\vee) \cap M$. Let $\tilde{\mathfrak{D}}$ be the polyhedral divisor with coefficients in $\tilde{N}_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} \tilde{N}$ given by

$$\tilde{\mathfrak{D}} = \operatorname{Conv}(0, \pi(v_0)) \cdot \{0\} + \operatorname{Conv}(0, \pi(v_1)) \cdot \{1\},$$

in the reflexive case, and by

$$\tilde{\mathfrak{D}} = \{\pi(v_0)\} \cdot \{0\} + \operatorname{Conv}(0, \pi(v_1)) \cdot \{1\}$$

in the skew case. Note that $e \in \operatorname{lin}(\sigma^\vee) \setminus \{0\}$ (see [Lie10a, Lemma 3.1]) and $\pi(v_0), \pi(v_1)$ are nonzero. We have a G_e -equivariant isomorphism $R \cong A[C, \tilde{\mathfrak{D}}]$. The induced homogeneous LNDs on $A[C, \tilde{\mathfrak{D}}]$ are horizontal and represented by the graphs

$$\begin{array}{ccccccc} \pi(v_0^-) & \pi(v_1^-) & & \pi(v_0^+) & \pi(v_1^+) & & \\ \cdots & \text{---} \circ & \text{---} \circ & \cdots & \cdots & \text{---} \circ & \text{---} \circ & \cdots \\ & 0 & 1 & & & 0 & 1 & \end{array}$$

In particular, Z is an affine spherical variety of type I which is homogeneous under G_e , according to Remark 3.5. So, Z is the closed G_e -orbit.

Case $C = \mathbb{P}^1$. By properness of \mathfrak{D} , every homogeneous element of degree $m \in \operatorname{lin}(\sigma^\vee) \cap M$ is invertible and so belongs to $\ker \partial_- \cap \ker \partial_+$. Hence I is G_e -stable. Furthermore, the \mathbb{T} -action on Z is transitive. This allows us to conclude. □

The following theorem provides the Luna–Vust description of the affine G_e -spherical variety X in terms of its AL-colorings $\mathfrak{D}_-, \mathfrak{D}_+$.

Theorem 3.7. *Let X be an affine G_e -spherical variety admitting a presentation as in Theorem 2.24. Then the following statements hold.*

- (i) *If $C = \mathbb{A}^1$ then X is a toroidal spherical variety with colored cone (σ, \emptyset) .*
- (ii) *If $C = \mathbb{P}^1$ then the colored cone of X is given by the following table.*

	<i>Reflexive case</i>	<i>Skew case</i>
B_- -action	$(\omega_-, \{D_{(0,0)}, D_{(1,v_1)}\})$	$(\omega_-, D_{(1,v_1)})$
B_+ -action	$(\omega_+, \{D_{(0,v_0)}, D_{(1,0)}\})$	$(\omega_+, D_{(1,0)})$

Proof. By Proposition 3.4 and Lemma 3.6, the closed orbit is contained in all colors of X in the case $C = \mathbb{P}^1$ and is not contained in any colors in the case $C = \mathbb{A}^1$. In the former case, one concludes by Lemma 1.11. \square

3.2. Horizontal and vertical G_e -invariant valuations.

As before, X is an affine embedding of type I of a spherical homogeneous space G_e/H with the associated colored cone $(\mathcal{C}, \mathcal{F}_Y)$. In this subsection, we provide the representation of $\mathbb{K}[X]$ by a polyhedral divisor and AL-colorings in terms of $(\mathcal{C}, \mathcal{F}_Y)$. For this, we examine the valuation cone of G_e/H .

3.8. Without loss of generality, we may suppose that $\mathbb{K}[G_e/H]$ is equal to the G_e -algebra $A[\mathbb{A}^1, \mathfrak{D}_{G_e/H}]$ defined in Remark 3.5. Thus, a variety Z is an affine embedding of G_e/H if and only if $\mathbb{K}[Z]$ is a G_e -stable normal affine subalgebra of $A[\mathbb{A}^1, \mathfrak{D}_{G_e/H}]$.

Inspired by [Tim08], we introduce the following terminology.

Definition 3.9. A non-trivial G_e -invariant valuation v on $\mathbb{K}(G_e/H)$ is called *horizontal* (or *central*) if v is trivial on the subfield $\mathbb{K}(t)$. Otherwise, v is said to be *vertical*. We denote by \mathcal{V}_{hor} (resp. $\mathcal{V}_{\text{vert}}$) the subset of horizontal (resp. vertical) valuations in the valuation cone \mathcal{V} of G_e/H .

We make the convention that $\mathcal{V}_{\text{hor}} \cap \mathcal{V}_{\text{vert}} = \{0\}$, i.e. the trivial valuation is both horizontal and vertical. For the next result, we emphasize that vectors in $N_{\mathbb{Q}}$ are regarded as primitive with respect to the valuation lattice $V = \text{Hom}(L, \mathbb{Z})$.

Lemma 3.10. *Consider the B_{\pm} -action on G_e/H . Then $\mathcal{V} = (\mathbb{Q}_{\geq 0}(\mp e))^{\vee}$. Further, denote*

$$S = \text{Cone}(-\rho_{\pm}(\mathcal{F}_0)),$$

where \mathcal{F}_0 is the set of colors of G_e/H . Then $\mathcal{V}_{\text{hor}} \cap (\mathcal{V} \setminus S) = e^{\perp} \setminus \{0\}$.

Proof. Let us show that $\mathcal{V} = \text{Cone}(\mp e)^{\vee}$ with respect to the B_{\pm} -action. For any primitive $\rho \in e^{\perp} \setminus \{0\}$, let us define the affine embedding $Z = \text{Spec } A[\mathbb{A}^1, \mathfrak{D}_{\rho}]$ of G_e/H via

$$\mathfrak{D}_{\rho} = \Delta_0^{\rho} \cdot \{0\} + \Delta_1^{\rho} \cdot \{1\},$$

where Δ_0^{ρ} and Δ_1^{ρ} are as in Theorem 2.24 with $\sigma = \text{Cone}(\rho)$. Then by Proposition 3.4, Z has a unique horizontal G_e -divisor D_{ρ} , hence $\rho = \varrho_{\pm}(D_{\rho})$ is contained in \mathcal{V} . This yields $e^{\perp} \subset \mathcal{V}_{\text{hor}}$. Using again Proposition 3.4, we may construct an affine embedding of G_e/H having a vertical G_e -divisor $D_{(v,\infty)}$ such that $\varrho_{\pm}(D_{(v,\infty)}) \in \{v \in N_{\mathbb{Q}} \mid \mp e(v) > 0\}$. Hence

$$\text{Cone}((\mp e)^{\vee}) = \text{Cone}(e^{\perp} \cup \varrho_{\pm}(D_{(v,\infty)})) \subset \mathcal{V}.$$

Since G_e/H is not horospherical, by [Kno91, Corollary 6.2] we obtain $\mathcal{V} = \text{Cone}(\mp e)^\vee$.

Take a primitive vector $v \in \mathcal{V} \setminus S$ and consider the colored cone $(\mathcal{C}, \mathcal{F}_Y)$ of G_e/H given by the following conditions:

- (1) \mathcal{F}_Y is the set of all colors of G_e/H ;
- (2) $\mathcal{C} = \mathbb{Q}_{\geq 0}v + \sum_{D \in \mathcal{F}_Y} \mathbb{Q}_{\geq 0}\varrho(D)$.

We note that \mathcal{C} is strongly convex, since $v \notin S$. Assume further that $v \notin e^\perp$. In this case $(\mathcal{C}, \mathcal{F}_Y)$ is an affine colored cone. Indeed, $\varrho(\mathcal{F}_Y) \cap \mathcal{V} = \emptyset$ by Proposition 3.4, and we set $m = 0$ in Definition 1.7. Let Z' be the embedding corresponding to $(\mathcal{C}, \mathcal{F}_Y)$. Then Z' is not toroidal and so admits a presentation $\mathbb{K}[Z'] = A[\mathbb{P}^1, \mathfrak{D}']$ as in Theorem 2.24. By 3.4, the only G_e -divisor D' on Z' is vertical and $\varrho(D') = v$. Thus, $v \in \mathcal{V}_{\text{vert}}$. The assertion follows. \square

Remark 3.11. The primitive vectors in $\mathcal{V} \setminus S$ are exactly the images of the G_e -divisors of all affine embeddings of G_e/H .

Definition 3.12. Given a vector $v \in N_{\mathbb{Q}} \setminus e^\perp$, we introduce the following function $\beta_v : N_{\mathbb{Q}} \rightarrow \mathbb{Q}$. Consider the linear projection $\pi_v : N_{\mathbb{Q}} \rightarrow \mathbb{Q} \cdot v$ along e^\perp . Then given $x \in N_{\mathbb{Q}}$, $\beta_v(x)$ is the unique rational number such that $\pi_v(x) = \beta_v(x)v$. We say that π_v is the projection defining β_v .

This notation is used in the following proposition that introduces the explicit construction of the AL-colorings corresponding to an affine embedding of G_e/H of type I. This is the inverse of Theorem 3.7.

Proposition 3.13. *Let X be an affine G_e -spherical variety of type I, whose open G_e -orbit is identified with the homogeneous space G_e/H . Let β_- be β_{v_0} and β_+ be either β_{v_1} in the reflexive case or $\beta_{v_0+v_1}$ in the skew case (cf. latter table in Proposition 3.4). Denote*

$$\mathcal{C}_{\pm}^{\text{vert}} = \left\{ \frac{v}{\beta_{\pm}(v)} \mid v \in \mathcal{C}(1), v(e) > 0 \right\}.$$

Let $\sigma \subset N_{\mathbb{Q}}$ be the strongly convex polyhedral cone equal to either

$$\sigma = \text{Cone}(\mathcal{C}(1) \cap e^\perp, \mathcal{C}_{\pm}^{\text{vert}}, \mathcal{C}_{\pm}^{\text{vert}} \pm v_0, \mathcal{C}_{\pm}^{\text{vert}} \mp v_1, \mathcal{C}_{\pm}^{\text{vert}} \pm v_0 \mp v_1)$$

in the reflexive case or

$$\sigma = \text{Cone}(\mathcal{C}(1) \cap e^\perp, \mathcal{C}_{\pm}^{\text{vert}}, \mathcal{C}_{\pm}^{\text{vert}} \mp v_1)$$

in the skew case. We have $\mathbb{K}[X] = A[C, \mathfrak{D}]$, where \mathfrak{D} is a σ -polyhedral divisor over $C = \mathbb{A}^1$, if $\mathcal{C} \subset e^\perp$, and over $C = \mathbb{P}^1$, otherwise. The non-trivial coefficients of \mathfrak{D} are defined as follows (see 3.5).

$$\Delta_0 = \Delta(G_e/H)_0 + \sigma, \quad \Delta_1 = \Delta(G_e/H)_1 + \sigma, \quad \text{and}$$

$$\Delta_\infty = \text{Conv} \left(\left\{ \frac{v - \pi_{\pm}(v)}{\beta_{\pm}(v)} \mid v \in \mathcal{C}(1) \setminus e^\perp \right\} \right) + \sigma \quad \text{if } C = \mathbb{P}^1,$$

where π_{\pm} is the projection defining β_{\pm} .

Proof. First, we note by [Kno91, Lemma 2.4] that $\mathcal{C}(1) \cap \mathcal{V}$ consists exactly of images in $N_{\mathbb{Q}}$ of the G_e -divisors of X , and elements of $\mathcal{C}(1) \cap e^\perp$ (resp. $\mathcal{C}(1) \setminus e^\perp$) correspond to horizontal (resp. vertical) G_e -divisors.

The G_e -algebra $\mathbb{K}[X] = A[C, \mathfrak{D}]$ admits a presentation as in Theorem 2.24, where \mathfrak{D} is a σ' -polyhedral divisor for some strongly convex polyhedral cone $\sigma' \subset N_{\mathbb{Q}}$. Using

Proposition 3.4 and Theorem 3.7, we remark that $\sigma \subset N_{\mathbb{Q}}$ is generated by $\sigma'(1)$ if $C = \mathbb{A}^1$ and by $\sigma'(1)^*$ and $\deg \mathfrak{D}$ if $C = \mathbb{P}^1$. Hence $\sigma = \sigma'$. Clearly, we have $\Delta_0 = \Delta(G/H)_0 + \sigma$ and $\Delta_1 = \Delta(G/H)_1 + \sigma$. For the case $C = \mathbb{P}^1$, the description of Δ_{∞} follows again from 3.4. \square

3.3. Classification of the type II.

In this subsection, we explain the connection between Demazure roots and affine colored cones for the type II. Let e be a nonzero vector of M . Note by [AL12, Proposition 3.8] an affine G_e -variety X is not of type I if and only if X is horospherical. The following result shows that an affine G_e -spherical X that is not of type I can be assumed to be of type II for a larger reductive group.

Lemma 3.14. *Let Ω be a quasi-affine G_e -horospherical homogeneous space. Assume that the induced \mathbb{T} -action on Ω is of complexity one. Then there exists an open subset $\tilde{\Omega}$ of the affine closure $\bar{\Omega} = \text{Spec } \mathbb{K}[\Omega]$, which is a homogeneous space under a reductive group $\tilde{G}_{\tilde{e}} = \text{SL}_2 \rtimes_{\tilde{e}} \tilde{\mathbb{T}}$ containing G_e with the following properties.*

- (a) *The torus \mathbb{T} is a subtorus of $\tilde{\mathbb{T}}$.*
- (b) *The $\tilde{\mathbb{T}}$ -action on $\tilde{\Omega}$ is of complexity 0.*
- (c) *$\tilde{\Omega}$ is an embedding of the G_e -spherical homogeneous space Ω .*

Moreover, the inclusion map $\Omega \rightarrow \tilde{\Omega}$ naturally identifies the affine G_e -embeddings of Ω with the affine $\tilde{G}_{\tilde{e}}$ -embeddings of $\tilde{\Omega}$.

Proof. First of all, $\mathbb{K}[\Omega]$ is finitely generated, so $\bar{\Omega} = \text{Spec } \mathbb{K}[\Omega]$ is an affine variety and $\text{codim}_{\bar{\Omega}}(\bar{\Omega} \setminus \Omega) \geq 2$. Since the SL_2 -action induced by G_e on $\bar{\Omega}$ is horospherical and horizontal, [AL12, Proposition 3.8] implies the existence of the $\tilde{G}_{\tilde{e}}$ -homogeneous space $\tilde{\Omega} \subset \bar{\Omega}$ that is a G_e -equivariant embedding of Ω .

Moreover, let X be an affine G_e -embedding of Ω . The $\tilde{G}'_{\tilde{e}}$ -action on X is constructed in [AL12, Proposition 3.8] by extending the torus $\mathbb{T} \subset G_e$ by the torus S commuting with G_e , see [AL12, Corollary 3.4]. The torus S corresponds to the grading defined by the decomposition into isotypic components with respect to the SL_2 -action. Hence it does not depend on the embedding X . Therefore, $\tilde{G}'_{\tilde{e}} = \tilde{G}_{\tilde{e}}$ and $\tilde{G}_{\tilde{e}}$ -action on X is the extension of $\tilde{G}_{\tilde{e}}$ -action on $\tilde{\Omega}$. \square

For the next paragraphs, we let X be an affine G_e -spherical variety of type II and assume that \mathbb{T} -action on X is faithful.

3.15. Since the torus \mathbb{T} acts on X with an open orbit, we may represent the M -graded algebra $\mathbb{K}[X]$ as the semigroup algebra

$$\mathbb{K}[\sigma^{\vee} \cap M] = \bigoplus_{m \in \sigma^{\vee} \cap M} \mathbb{K}\chi^m,$$

where $\sigma \subset N_{\mathbb{Q}}$ is a strongly convex polyhedral cone. For any $m \in \sigma^{\vee} \cap M$ the character $\chi^m : \mathbb{T} \rightarrow \mathbb{G}_m$ is identified with a regular function on X . Recall that the set of *semi-simple roots* [Oda88, Section 3.4] of σ is

$$\text{Rt}_{\text{ss}}(\sigma) = \text{Rt}(\sigma) \cap (-\text{Rt}(\sigma)).$$

By [AL12, Theorem 2.7], we have $e \in \text{Rt}_{\text{ss}}(\sigma)$. Denoting by ρ_{\pm} the distinguished ray of $\pm e$, the \mathfrak{sl}_2 -triplet on $\mathbb{K}[X]$ corresponding to the G_e -action on X is $\{\partial_-, \delta, \partial_+\}$, where $\partial_{\pm}(\chi^m) = \langle m, \rho_{\pm} \rangle \chi^{m \pm e}$ for all $m \in \sigma^{\vee} \cap M$, and $\delta = [\partial_+, \partial_-]$.

Lemma 3.16. *The following assertions hold.*

- (i) *The lattice L_{\pm} of B_{\pm} -weights of $\mathbb{K}(X)$ is the sublattice of M generated by $\tau_{\pm} \cap M$, where $\tau_{\pm} \subset \sigma^{\vee}$ is the dual facet of the ray $\rho_{\pm} \subset \sigma$.*
- (ii) *The open B_{\pm} -orbit on X is $X_{B_{\pm}} = \text{Spec } \mathbb{K}[\rho_{\pm}^{\vee} \cap M]$.*
- (iii) *The open G_e -orbit on X is the toric \mathbb{T} -variety X_{Σ} associated to the fan $\Sigma = \{\rho_-, \{0\}, \rho_+\}$.*

Proof. (i) This follows from the equality $\ker \partial_{\pm} = \mathbb{K}[\tau_{\pm} \cap M]$.

(ii) $X_{B_{\pm}} \cong \mathbb{A}^1 \times \mathbb{G}_m^{n-1}$, where $n = \dim \mathbb{T}$, is an open B_{\pm} -stable subset of X . It consists of two \mathbb{T} -orbits, namely $\{0\} \times \mathbb{G}_m^{n-1}$ and $\mathbb{G}_m \times \mathbb{G}_m^{n-1} \cong \mathbb{T}$. Since a U_{\pm} -orbit, which is an affine line, cannot be contained in neither \mathbb{T} -orbit, one concludes.

(iii) By the above step, the open B_{\pm} -orbit corresponds to the fan $\{\{0\}, \rho_{\pm}\}$, hence the open G_e -orbit contains X_{Σ} . So it remains to show that X_{Σ} is SL_2 -stable. The coordinate ring of X_{Σ} is $\mathbb{K}[\Lambda \cap M]$, where $\Lambda = \rho_-^{\vee} \cap \rho_+^{\vee} \subset M_{\mathbb{Q}}$. Since $\mathbb{K}[X_{\Sigma}]$ is SL_2 -stable, it remains to show that the complement $\text{Spec}(\mathbb{K}[\Lambda \cap M]) \setminus X_{\Sigma}$ is also SL_2 -stable. The ideal of the complement

$$I = \bigoplus_{m \in (\Lambda \setminus \text{lin}(\Lambda)) \cap M} \mathbb{K}\chi^m \subset \mathbb{K}[\Lambda^{\vee} \cap M]$$

is ∂_{\pm} -stable, which proves the assertion. \square

Remark 3.17. Given an affine horospherical embedding X , the associated colored cone is $(\mathcal{C}, \mathcal{F})$, where \mathcal{F} is the set of all colors of X , see Definition 1.7. In particular, for any affine embedding of X_{Σ} the associated colored cone contains exactly one color.

The main result of this subsection is the following proposition. It can also be obtained from [AKP13, Theorem 1.1].

Proposition 3.18. *Let X be an affine G_e -spherical variety of type II. Assume that the induced \mathbb{T} -action on X is faithful.*

- (i) *Assume that X is the toric \mathbb{T} -variety corresponding to the strongly convex cone $\sigma \subset N_{\mathbb{Q}}$. Let ω_{\pm} be the image of the cone*

$$\sum_{\rho \in \sigma(1) \setminus \{\rho_{\pm}\}} \mathbb{Q}_{\geq 0} \cdot \rho \subset \sigma$$

under the projection $N_{\mathbb{Q}} \rightarrow (V_{\pm})_{\mathbb{Q}} = \text{Hom}(L_{\pm}, \mathbb{Q}) = (N/\langle \rho_{\pm} \rangle)_{\mathbb{Q}}$, and $D_{\rho_{\pm}}$ be the \mathbb{T} -divisor on X corresponding to the ray ρ_{\pm} . Then the associated colored cone of X in $(V_{\pm})_{\mathbb{Q}}$ with respect to the B_{\pm} -action is $(\omega_{\pm}, D_{\rho_{\pm}})$.

- (ii) *Conversely, assume that X is an affine embedding of the G_e -homogeneous space X_{Σ} (see 3.16) given by a colored cone (\mathcal{C}, D_{\pm}) , where D_{\pm} is the color of X for the B_{\pm} -action. Then X is equal to the toric \mathbb{T} -variety $\text{Spec } \mathbb{K}[\mathcal{C}^{\vee} \cap \rho_{\pm}^{\vee} \cap M]$, where $\rho_{\pm} \in N$ is the primitive vector corresponding to the \mathbb{T} -divisor D_{\pm} .*

Proof. (i) By Lemma 3.16, the G_e -divisors of X correspond to elements of $\sigma(1) \setminus \{\rho_-, \rho_+\}$ and the image of the unique color (with respect to B_{\pm}) is ρ_{\pm} .

Assertion (ii) follows directly from (i). \square

4. DEMAZURE ROOTS AND AFFINE SPHERICAL VARIETIES

The existence of normalized \mathbb{G}_a -actions on an affine spherical variety imposes restrictive conditions on its geometric structure. We study some of these conditions in the

framework of the Luna–Vust theory. In particular, we define the notion of Demazure roots of an affine spherical embedding X (see 4.3), which describes the normalized \mathbb{G}_a -actions on X . Section 4.2 is devoted to the example of affine G_e -spherical varieties, where G_e is the reductive group $\mathrm{SL}_2 \rtimes_e \mathbb{T}$.

4.1. General case.

We denote by G a connected reductive group and fix a Borel subgroup $B \subset G$. We also let U be the unipotent radical of B . We begin with a preliminary lemma.

Lemma 4.1. *Let Z be a quasi-affine variety endowed with a G -action. Let ∂ be an LND on $\mathbb{K}[Z]$. Then the \mathbb{G}_a -action corresponding to ∂ is normalized of degree χ if and only if*

$$g \cdot \partial(f) = \chi(g) \cdot \partial(g \cdot f)$$

for all $f \in \mathbb{K}[Z]$ and $g \in G$. If the latter condition holds, then ∂ sends B -eigenvectors into B -eigenvectors. Furthermore, if Z contains an open B -orbit and ∂ is nonzero, then ∂ is uniquely determined by the character χ up to the multiplication by a nonzero constant.

Proof. See [FKZ13, Section 1.1] for the existence and properties of the LND ∂ . The first claims are obtained similarly to the proof of [FZ05, Lemma 2.2]. Moreover, for any $b \in B$ and B -eigenvector $f \in \mathbb{K}[Z]$ with B -weight χ_f , one has

$$b \cdot \partial(f) = \chi(b) \cdot \partial(b \cdot f) = (\chi \cdot \chi_f)(b) \cdot \partial(f).$$

Hence, ∂ sends B -eigenvectors into B -eigenvectors.

Assume that Z contains an open B -orbit. Passing to the normalization, we may suppose that Z is normal. Since the semi-simple G -module $\mathbb{K}[Z]$ has no multiplicities, the subalgebra $\mathbb{K}[Z]^U$ is a ∂ -stable normal finitely generated semi-group algebra. Furthermore, the restriction ∂' of ∂ on $\mathbb{K}[Z]^U$ is a homogeneous LND. Since every simple submodule of $\mathbb{K}[X]$ is generated by $G \cdot f$ for some B -weight $f \in \mathbb{K}[X]$, ∂ is uniquely determined by ∂' . One concludes by [Lie10a, Corollary 2.8]. \square

In the sequel, we fix an affine embedding X of a spherical homogeneous space G/H given by a colored cone $(\mathcal{C}, \mathcal{F}_Y)$. We denote by \mathcal{F}_0 the set of colors of X and by Γ the cone generated by $\varrho(\mathcal{F}_0) \cup \mathcal{C}$. We recall that Γ^\vee is the cone of B -weights of $\mathbb{K}[X]$ (see Lemma 1.11).

4.2. An embedding Z of G/H is called *elementary* if Z has two orbits so that the complement of the open orbit is a divisor. In particular, every elementary embedding is smooth. There is a natural bijection between non-trivial normalized discrete G -invariant valuations on $\mathbb{K}(G/H)$ and elementary embeddings of G/H (see [BP87, 2.2]).

Let $v \in \Gamma(1)$. We say that v *corresponds to a G -divisor* of X if there exists a G -divisor $D \subset X$ such that $v = \varrho(D)$. In the sequel, we make the following convention. If v corresponds to a G -divisor of X , then X_v denotes the associated elementary embedding of G/H given by the colored cone $(\mathbb{Q}_{\geq 0} \cdot v, \emptyset)$. Otherwise, $v \in \mathbb{Q}_{\geq 0} \cdot \varrho(D_0)$ for some $D_0 \in \mathcal{F}_0$ (see [Kno91, Lemma 2.4]) and we let $X_v = G/H$.

Let us introduce the main definition of this subsection.

Definition 4.3. A character χ of G is called a *Demazure root* of X if $\chi \in \mathrm{Rt}(\Gamma)$ and the following holds. The embedding X_v , where $v = \rho_\chi$ is the distinguished ray of χ

(see 2.10), admits a normalized \mathbb{G}_a -action of degree χ and is homogeneous under the induced $\mathbb{G}_a \rtimes_\chi G$ -action.

We denote by $\text{Rt}(X)$ the set of Demazure roots of X . A root $\chi \in \text{Rt}(X)$ is said to be *exterior* if ρ_χ corresponds to a G -divisor. Otherwise, χ is called *interior*. We denote by $\text{Rt}_{\text{ext}}(X)$ (resp. $\text{Rt}_{\text{int}}(X)$) the set of exterior (resp. interior) roots of X . Clearly, we have

$$\text{Rt}_{\text{ext}}(X) \cap \text{Rt}_{\text{int}}(X) = \emptyset \text{ and } \text{Rt}(X) = \text{Rt}_{\text{ext}}(X) \cup \text{Rt}_{\text{int}}(X).$$

4.4. Two nonzero LNDs ∂, ∂' on $\mathbb{K}[X]$ are called equivalent if $\partial = \lambda \cdot \partial'$ for some $\lambda \in \mathbb{K}^*$. We denote by $\text{LND}_G(X)$ the set of equivalence classes of nonzero LNDs on $\mathbb{K}[X]$ corresponding to normalized \mathbb{G}_a -actions. Since $\deg \partial = \deg \partial'$ for equivalent ∂ and ∂' , the degree is well defined on $\text{LND}_G(X)$. It also coincides with the degree of the homogeneous LND on $\mathbb{K}[X]^U$ induced by any $\partial' \in [\partial]$.

The following theorem allows to reduce the classification of normalized \mathbb{G}_a -actions on X to the case when X is either quasi-affine homogeneous or elementary, and the induced action of $\mathbb{G}_a \rtimes_\chi G$ on X is transitive.

Theorem 4.5. *The map*

$$(6) \quad \iota : \text{LND}_G(X) \rightarrow \text{Rt}(X), \quad [\partial] \mapsto \deg \partial$$

is well defined and bijective. Furthermore, let $\partial \in \text{LND}_G(X)$ and $\chi = \deg \partial \in \text{Rt}(X)$ have the distinguished ray v . Then the following hold.

- (i) *The open orbit of the induced $\mathbb{G}_a \rtimes_\chi G$ -orbit on X is X_v .*
- (ii) *We have $-v \in \mathcal{V}$.*
- (iii) *The kernel $\ker \partial$ is a finitely generated subalgebra.*

Proof. Consider $[\partial] \in \text{LND}_G(X)$. Let us show that $\chi = \deg \partial \in \text{Rt}(X)$. By Lemmas 1.11, 4.1, and [Lie10a, Theorem 2.7], we have $\chi \in \text{Rt}(\Gamma)$. Choose a B -eigenvector $f \in \mathbb{K}[X]$ such that $\chi_f \in \text{rel.int.}(\Gamma \cap \rho_\chi^\perp) \cap L$, and let $B.x$ be the open B -orbit of X . Since $f \in \ker \partial$, the subset $X_f = X \setminus \text{div}(f)$ is stable under the \mathbb{G}_a -action induced by ∂ .

Assume first that $v = \rho_\chi \in \Gamma(1)$ does not correspond to a G -divisor of X . Then we may consider the non-empty subset of colors

$$\mathcal{F}' = \{D' \in \mathcal{F}_0 \mid \varrho(D') = v\}.$$

Using the relation

$$\text{div } f = \sum_{D \in \mathcal{F}_0 \cup \mathcal{P}} \langle \chi_f, \varrho(D) \rangle \cdot D,$$

where \mathcal{P} is the set of G -divisors of X , we obtain the equality

$$X_f = X \setminus \bigcup_{\mathcal{P} \cup \mathcal{F}_0 \setminus \mathcal{F}'} D.$$

The complement to $(G/H) \cap X_f$ in X_f is of codimension at least two, hence the subset

$$Z = \mathbb{G}_a \star (X_f \setminus (G/H))$$

is \mathbb{G}_a -stable and of codimension at least one. Thus, $G/H = G \cdot (X_f \setminus Z)$ is also \mathbb{G}_a -stable. Hence $\chi \in \text{Rt}_{\text{int}}(X)$ and (i) is proven for this case.

Now assume that $v = \rho_\chi \in \Gamma(1)$ corresponds to a G -divisor D_v of X . Then

$$X_f = X \setminus \bigcup_{(\mathcal{P} \cup \mathcal{F}_0) \setminus \{D_v\}} D.$$

So, the open subset $G \cdot X_f$ is \mathbb{G}_a -stable and its unique G -divisor is D_v .

Let us prove by contradiction that $\mathbb{G}_a \star D_v$ is dense in X . If $\mathbb{G}_a \star D_v$ is not dense, then G/H is \mathbb{G}_a -stable. Remark that the dual cone Γ_0 of the B -weights of $\mathbb{K}[G/H]$ is the subcone of Γ generated by $\varrho(\mathcal{F}_0)$. By considering the induced LND on $\mathbb{K}[G/H]^U$ (see 4.1), which has same degree χ , we have $\chi \in \text{Rt}(\Gamma_0)$. Let v' be the distinguished ray of χ with respect to Γ_0 . The inclusion

$$\mathbb{K}[X]^U \cap \mathbb{K}(X)^\partial \subset \mathbb{K}[G/H]^U \cap \mathbb{K}(X)^\partial$$

yields the inclusion of facets $v^\perp \cap \Gamma^\vee \subset v'^\perp \cap \Gamma_0^\vee$. Hence $v^\perp = v'^\perp$. Since v, v' are primitive vectors and Γ is strongly convex, we have $v = v' \in \Gamma(1) \cap \Gamma_0(1)$. Therefore v does not correspond to a G -divisor of X , a contradiction. Hence $\mathbb{G}_a \star D_v$ is dense in X .

Let $G.y$ be the orbit which closure in $G \cdot X_f$ is D_v . Such an orbit $G.y$ exists due to the fact that X has a finite number of orbits. Then the subset

$$Z_0 = G \cdot X_f \setminus (G/H \cup G.y)$$

is closed of codimension at least two. So, $Z_1 = \overline{\mathbb{G}_a \star Z_0}$ is a $\mathbb{G}_a \rtimes_\chi G$ -stable subset of codimension at least one. Hence $Z_1 \cap G/H = \emptyset$. If $Z_1 \cap G.y \neq \emptyset$, then Z_1 contains D_v , yielding a contradiction. Therefore, $Z_1 = Z_0$ and X_v is \mathbb{G}_a -stable. Since D_v is not \mathbb{G}_a -stable, $\mathbb{G}_a \rtimes_\chi G$ acts homogeneously on X_v and $\chi \in \text{Rt}_{\text{ext}}(X)$. This proves (i) and shows that ι in (6) is well defined. Note that the injectivity of ι follows from Lemma 4.1 and the surjectivity is a consequence of (i).

(ii) We define the discrete G -invariant valuation w on $\mathbb{K}(G/H)$ by letting

$$w(f) = -\max\{i \in \mathbb{N} \mid \partial^i(f) \neq 0\}$$

for every nonzero $f \in \mathbb{K}[X_v]$. Further, if f is a B -eigenvector, then $w(f) = \langle \chi_f, -v \rangle$. Hence $w = -v \in \text{lin}.\mathcal{V}$.

(iii) This follows from the fact that the $(\ker \partial)^U$ is finitely generated (see [Kur03, Theorem 1.2] and [Lie10a, Corollary 2.11]) and that $\ker \partial = G \cdot (\ker \partial)^U$. \square

By the previous result we may introduce the following definition.

Definition 4.6. A non-trivial normalized \mathbb{G}_a -action of degree χ on X is said to be *exterior* (resp. *interior*) if $\chi \in \text{Rt}_{\text{ext}}(X)$ (resp. $\chi \in \text{Rt}_{\text{int}}(X)$)

As a direct consequence of Theorem 4.5, we have the following results.

Corollary 4.7. *If G is semi-simple, then $\text{Rt}(X) = \emptyset$. If \mathcal{V} is strongly convex, then $\text{Rt}_{\text{ext}}(X) = \emptyset$.*

Proof. This follows from the fact that $\text{Rt}(X) \subset \text{Rt}(\Gamma)$ and from the part (ii) of Theorem 4.5. \square

Corollary 4.8. *If X has three orbits including a fixed point y , then $\text{Rt}_{\text{ext}}(X) = \emptyset$.⁵*

Proof. Take $\chi \in \text{Rt}_{\text{ext}}(X)$. By Theorem 4.5, X admits a normalized \mathbb{G}_a -action on X that induces a transitive $\mathbb{G}_a \rtimes_\chi G$ -action on X_v , where $v \in \mathcal{C}(1) \cap \mathcal{V}$. Therefore, $X \setminus X_v$ is \mathbb{G}_a -stable and y is the unique fixed point for the \mathbb{G}_a -action. This yields a contradiction (see [Bia73]). \square

⁵We are thankful for valuable suggestions to Hanspeter Kraft and other participants of the seminar “Algebra and Geometry” at the Mathematisches Institut, Basel Universität.

Example 4.9. Consider the spherical action of $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ on the variety $X = \mathrm{M}_{2 \times 2}$ of 2×2 matrices via the relation $(P, Q) \cdot M = PMQ^{-1}$ for all $(P, Q) \in G$ and $M \in X$. Then X has three orbits:

$$G \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the first is the fixed point. Hence by Corollary 4.8, every exterior normalized \mathbb{G}_a -action on X is trivial.

In the following paragraph, we recall the construction of the odd symplectic Grassmannian (cf [Mi07]). This construction provides examples of non-trivial exterior normalized \mathbb{G}_a -actions in the projective situation.

Example 4.10. Let E be a \mathbb{K} -vector space of dimension $2r + 1$. Consider a skew form $\omega \in \wedge E^*$ of rank $2r$, where E^* is the dual vector space of E . The closed subgroup of $\mathrm{GL}(E)$ preserving the form ω is called the *odd symplectic group* Sp_{2r+1} . Up to an appropriate basis of E , this algebraic group is isomorphic to a semi-direct product $\mathbb{G}_a^{2r} \rtimes (\mathrm{Sp}_{2r} \times \mathbb{G}_m)$, where $\mathrm{Sp}_{2r} \subset \mathrm{GL}_{2r}$ is the classical symplectic group. Recall that a linear subspace $F \subset E$ is said to be *isotropic* (with respect to ω) if $\omega(v_1, v_2) = 0$ for any $v_1, v_2 \in F$. To specify the latter property we will write $\omega(F) = 0$. Denote by $\mathrm{Gr}_d(E)$ the Grassmannian of d -dimensional linear subspaces of E . Then the *odd symplectic Grassmannian* is defined as the closed irreducible subset

$$X = \{F \in \mathrm{Gr}_d(E) \mid \omega(F) = 0\}.$$

Assume that $d \leq r$. The algebraic group Sp_{2r+1} acts naturally on X . Let $G = \mathrm{Sp}_{2r} \times \mathbb{G}_m \subset \mathrm{Sp}_{2r+1}$, then the variety X is G -spherical. By [Mi07, Proposition 4.3], the variety has an open Sp_{2r+1} -orbit \mathcal{O} , which decomposes in two G -orbits. Hence the induced \mathbb{G}_a^{2r} -action on X is non-trivial. Furthermore, \mathcal{O} is the total space of a line bundle over the symplectic Grassmannian Z of d -linear subspaces of \mathbb{K}^{2r} , whose fibers coincide with the \mathbb{G}_a^{2r} -orbits and also with the closures of the \mathbb{G}_m -orbits in \mathcal{O} . The closed G -orbit in \mathcal{O} corresponds to the zero section and is therefore isomorphic to Z . In particular, every one-parameter unipotent subgroup $H \subset \mathbb{G}_a^{2r}$ yields a vertical \mathbb{G}_a -action on \mathcal{O} normalized by \mathbb{G}_m .

The set $\mathrm{Rt}(X)$ of Demazure roots of an affine spherical variety X is defined in geometric terms. A natural question is to know whether $\mathrm{Rt}(X)$ is described in combinatorial terms. More precisely, we propose the following problem.

Question 4.11. For any affine spherical variety X , is the set $\mathrm{Rt}(X)$ equal to the intersection of a finite union of polyhedra in $L_{\mathbb{Q}}$ with the lattice L ?

4.2. Description for the reductive group G_e .

Let X be an affine G_e -spherical variety with associated colored cone $(\mathcal{C}, \mathcal{F})$. We assume that $e \neq 0$. The aim of this subsection is to provide an explicit description of the set $\mathrm{Rt}(X)$ of the Demazure roots of X in terms of the pair $(\mathcal{C}, \mathcal{F})$, see Theorem 4.18.

4.12. Let ∂ be the LND on $\mathbb{K}[X]$ corresponding to a non-trivial normalized \mathbb{G}_a -action of degree χ^θ , where $\theta \in M$ is nonzero. Equivalently, ∂ is nonzero, homogeneous of degree θ with respect to the M -grading of $\mathbb{K}[X]$, and satisfies $[\partial, \partial_-] = [\partial, \partial_+] = 0$, where ∂_\pm is the LND corresponding to the U_\pm -action on X .

Applying Lemma 4.1 to the Borel subgroups B_- and B_+ of G_e , we immediately obtain the following result.

Corollary 4.13. *Assume that X is of type I. Without loss of generality, we may suppose that $\mathbb{K}[X]$ admits a presentation as in 2.24. Then in the notation of Section 3 and 4.12, we have $\theta \in \text{Rt}(\omega_-) \cap \text{Rt}(\omega_+)$, and there exists a facet $\varsigma_{\pm} \subset \omega_{\pm}^{\vee}$ such that*

$$\ker \partial_{\pm} \cap \ker \partial = \bigoplus_{m \in \varsigma_{\pm} \cap L} \mathbb{K} \varphi_m^{\pm} \chi^m.$$

Proof. By Lemma 4.1, the restriction of ∂ to $\ker \partial_{\pm} = \mathbb{K}[X]^{U_{\pm}}$ is well defined, non-trivial, and homogeneous. Since $\ker \partial_{\pm}$ is a semigroup algebra, we conclude by [Lie10a, Lemma 2.6]. \square

The following theorem gives examples of affine spherical homogeneous spaces admitting a non-trivial normalized \mathbb{G}_a -action.

Theorem 4.14. *Let X be a G_e -spherical homogeneous space of type I. Consider the representation $\mathbb{K}[X] = A[\mathbb{A}^1, \mathfrak{D}]$ as in Remark 3.5, where $\mathfrak{D} = \mathfrak{D}_{G_e/H}$. Then X admits a normalized \mathbb{G}_a -action if and only if X is reflexive and v_0, v_1 are linearly independent. Under this condition, all normalized \mathbb{G}_a -actions are horizontal, and we have*

$$\text{Rt}(X) = \text{Rt}_{\text{int}}(X) = \{\theta \in L \mid v_0(\theta) = v_1(\theta) = 1 \text{ or } v_0(\theta) = v_1(\theta) = -1\}.$$

Proof. Let ∂ be the LND corresponding to a non-trivial normalized \mathbb{G}_a -action on X . Denote by θ the degree of ∂ , and by ϖ the weight cone of $\ker \partial$. By [Lie10a, Lemma 3.1], ∂ is horizontal with respect to the M -grading of $\mathbb{K}[X]$.

Assume that X is skew. Then the quasifan $\Lambda(\mathfrak{D})$ (see 2.2) has exactly two maximal elements. By Theorem 2.15, they are exactly ω_-^{\vee} and ω_+^{\vee} , hence ϖ coincides with one of them. But this contradicts Corollary 4.13. Thus, we may suppose that X is reflexive and $\Lambda(\mathfrak{D})$ has at least three maximal cones. The latter condition is equivalent to v_0, v_1 being linearly independent. In fact, in this case $\Lambda(\mathfrak{D})$ has four maximal cones, and ω_- , ω_+ , ϖ are among them.

By Theorem 2.15, the LND ∂ can be given by a triple $(\lambda, \mathfrak{D}_{\bullet}, \theta)$, where we let $\lambda = 1$ for simplicity and where the colored polyhedral divisor is given by

$$\mathfrak{D}_{\bullet} = (\mathfrak{D}, \{\mu_z \mid z \in \mathbb{A}^1\}, z_0).$$

Since ω_- and ω_+ correspond to diagrams

$$\begin{array}{ccc} v_0 & 0 & \\ \cdots \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots & & \cdots \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \\ 0 & 1 & 0 \quad v_1 \\ & & 0 \quad 1 \end{array}$$

the two possible cases remaining for ϖ are

$$\begin{array}{ccc} 0 & 0 & \\ \cdots \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots & & \cdots \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \\ 0 & 1 & v_0 \quad v_1 \\ & & 0 \quad 1 \end{array}$$

where the marked point z_0 is not yet known.

Case 1. The coloration \mathfrak{D}_{\bullet} is represented by one of diagrams

$$\begin{array}{ccccccc} 0 & 0 & & 0 & 0 & & 0 & 0 & 0 \\ \cdots \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_0 = 0 & 1 & & 0 & z_0 = 1 & & 0 & 1 & z_0 \end{array}$$

The LND ∂ is given by the relation

$$\partial(Q\chi^m) = Q'\chi^{m+\theta} \text{ for any } Q \in \mathbb{K}(t), \chi \in M.$$

Let us study the conditions $[\partial, \partial_+] = [\partial, \partial_-] = 0$. Recall that

$$\begin{aligned} \partial_-(Q\chi^m) &= (v_0(m)Q + tQ')\chi^{m-e}, \text{ for any } Q \in \mathbb{K}(t), \chi \in M, \\ \partial_+(Q\chi^m) &= (v_1(m)Q + (t-1)Q')\chi^{m+e} \text{ for any } Q \in \mathbb{K}(t), \chi \in M. \end{aligned}$$

Then

$$\begin{aligned} \partial \circ \partial_-(Q\chi^m) &= (v_0(m)Q' + Q' + tQ'')\chi^{m+\theta-e}, \\ \partial_- \circ \partial(Q\chi^m) &= (v_0(m+\theta)Q' + tQ'')\chi^{m+\theta-e}, \\ \partial \circ \partial_+(Q\chi^m) &= (v_1(m)Q' + Q' + (t-1)Q'')\chi^{m+\theta+e}, \\ \partial_+ \circ \partial(Q\chi^m) &= (v_1(m+\theta)Q' + (t-1)Q'')\chi^{m+\theta+e}. \end{aligned}$$

The conditions $[\partial, \partial_-] = 0$ and $[\partial, \partial_+] = 0$ are therefore equivalent $v_0(\theta) = 1$ and $v_1(\theta) = 1$ respectively.

Case 2. The set of AL-colors $\{\mu_z \mid z \in \mathbb{A}^1\}$ is $\{v_0, v_1\}$. Since $\varsigma_{\pm} = \varpi \cap \omega_{\pm}^{\vee}$ is a facet that separates full-dimensional cones ϖ and ω_{\pm}^{\vee} in σ^{\vee} , we have $\varsigma_{\pm} \cap \text{rel.int.}(\sigma^{\vee}) \neq \emptyset$. This implies that

$$\ker \partial = \bigoplus_{m \in \varpi \cap L} \mathbb{K}t^{-v_0(m)}(t-1)^{-v_1(m)},$$

and therefore $z_0 \in \{0, 1\}$ by Remark 2.16. By applying the formula from Remark 2.16, one can see that in both cases $z_0 = 0$ and $z_0 = 1$ the derivation ∂ is the same. namely for any $Q \in \mathbb{K}(t), m \in M$

$$\partial(Q\chi^m) = \left(\left(\frac{v_0(m)}{t} + \frac{v_1(m)}{t-1} \right) Q + Q' \right) \gamma \chi^{m+\theta} \text{ where } \gamma = t^{-v_0(\theta)}(t-1)^{-v_1(\theta)}.$$

Let us study the conditions $[\partial, \partial_-] = [\partial, \partial_+] = 0$. Using $\gamma' = -v_0(\theta)\frac{\gamma}{t} - v_1(\theta)\frac{\gamma}{t-1}$, we have

$$\begin{aligned} \partial \circ \partial_-(t) &= \partial(t\chi^{-e}) = \frac{t}{t-1} \gamma \chi^{\theta-e}, \\ \partial_- \circ \partial(t) &= \partial_-(\gamma \chi^{\theta}) = -v_1(\theta) \frac{t}{t-1} \gamma \chi^{\theta-e}, \\ \partial \circ \partial_+(t) &= \partial((t-1)\chi^e) = \frac{t-1}{t} \gamma \chi^{\theta+e}, \\ \partial_+ \circ \partial(t) &= \partial_+(\gamma \chi^{\theta}) = -v_0(\theta) \frac{t-1}{t} \gamma \chi^{\theta+e}, \end{aligned}$$

The conditions $\partial \circ \partial_+(t) = \partial_+ \circ \partial(t)$ and $\partial \circ \partial_-(t) = \partial_- \circ \partial(t)$ are therefore equivalent $v_0(\theta) = -1$ and $v_1(\theta) = -1$ respectively. Thus, $\gamma = t(t-1)$ and

$$\partial(Q\chi^m) = ((v_0(m)(t-1) + v_1(m)t)Q + t(t-1)Q')\chi^{m+\theta}.$$

It is easy to check that $\partial \circ \partial_-(\chi^m) = \partial_- \circ \partial(\chi^m)$ and $\partial \circ \partial_+(\chi^m) = \partial_+ \circ \partial(\chi^m)$ for any $m \in M$. Therefore, ∂ commutes with ∂_- and ∂_+ .

Finally, from cases 1 and 2 we obtain the condition $v_0(\theta) = v_1(\theta) = \pm 1$, which implies that θ is the Demazure root of the domain of \mathfrak{D}_{\bullet} . Now the proof of the theorem is completed. \square

Let us give an example of a non-trivial interior normalized \mathbb{G}_a -action.

Example 4.15. Assume that $M = N = \mathbb{Z}^2$ and $M_{\mathbb{Q}} = N_{\mathbb{Q}} = \mathbb{Q}^2$. The pairing between M and N is given by the usual scalar product. Consider the polyhedral divisor

$$\mathfrak{D} = \text{Conv}(0, v_0) \cdot \{0\} + \text{Conv}(0, v_1) \cdot \{1\}$$

over the affine line \mathbb{A}^1 , where $v_0 = (-1, 0)$ and $v_1 = (0, -1)$. An easy computation shows that $A = A[\mathbb{A}^1, \mathfrak{D}] = \mathbb{K}[u_1, u_2, u_3, u_4]$, where $u_1 = \chi^{(-1,0)}$, $u_2 = \chi^{(0,-1)}$, $u_3 = (t-1)\chi^{(0,1)}$, and $u_4 = t\chi^{(1,0)}$ satisfy the irreducible relation $u_1u_4 - u_2u_3 - 1$. Hence $X = \text{Spec } A$ is identified with the hypersurface $\mathbb{V}(x_1x_4 - x_2x_3 - 1) \subset \mathbb{A}^4$. The action of the torus $\mathbb{T} = (\mathbb{K}^*)^2$ on the coordinates is given by the formula

$$(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1 \cdot x_1, t_2 \cdot x_2, t_2^{-1} \cdot x_3, t_1^{-1} \cdot x_4),$$

where $(t_1, t_2) \in \mathbb{T}$. Consider the vector $e = (-1, 1)$. Then we have $\langle v_0, e \rangle = 1$ and $\langle v_1, e \rangle = -1$. The corresponding LNDs are given by

$$\partial_- = u_2 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_3} \quad \text{and} \quad \partial_+ = u_1 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_4}.$$

Thus, the action of the algebraic group $G_e = \text{SL}_2 \rtimes_e \mathbb{T}$ is defined by the formula

$$g \cdot (x_1, x_2, x_3, x_4) = (dt_1x_1 + ct_2x_2, bt_1x_1 + at_2x_2, dt_2^{-1}x_3 + ct_1^{-1}x_4, bt_2^{-1}x_3 + at_1^{-1}x_4),$$

where $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (t_1, t_2) \right) \in G_e$. In particular, we have

$$g \cdot (1, 0, 0, 1) = (dt_1, bt_1, ct_1^{-1}, at_1^{-1}).$$

The isotropy subgroup of $(1, 0, 0, 1)$ is equal to

$$H = \left\{ \left(\begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, (t_1, t_2) \right) \mid t_1, t_2 \in \mathbb{K}^* \right\},$$

and we have a G_e -isomorphism $X \simeq G_e/H$. Let us describe the non-trivial normalized \mathbb{G}_a -actions on X . By Theorem 4.14, we observe that $\text{Rt}_{\text{int}}(X) = \{(1, 1), (-1, -1)\}$. The associated LNDs to the vectors $(-1, -1)$, $(1, 1)$ are given respectively by

$$u_3 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_2} \quad \text{and} \quad u_1 \frac{\partial}{\partial u_3} + u_2 \frac{\partial}{\partial u_4},$$

and the associated \mathbb{G}_a -actions by

$$\lambda \star (x_1, x_2, x_3, x_4) = (x_1 + \lambda x_3, x_2 + \lambda x_4, x_3, x_4)$$

and

$$\lambda \star (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3 + \lambda x_1, x_4 + \lambda x_2)$$

for any $\lambda \in \mathbb{G}_a$.

Let us describe the $\mathbb{G}_a \rtimes_{\theta} G_e$ -homogeneous space $X = X_{\rho}$ associated to the root $\theta \in \text{Rt}_{\text{int}}(X)$ with the distinguished ray ρ . If $\theta = (-1, -1)$, then we have

$$(\lambda, g) \cdot (1, 0, 0, 1) = \lambda \star (g \cdot (1, 0, 0, 1)) = (dt_1 + \lambda ct_1^{-1}, bt_1 + \lambda at_1^{-1}, ct_1^{-1}, at_1^{-1}).$$

The isotropy subgroup of $(1, 0, 0, 1)$ for the $\mathbb{G}_a \rtimes_{\theta} G_e$ -action is

$$H_1 = \left\{ \left(-bt_1, \begin{pmatrix} t_1 & b \\ 0 & t_1^{-1} \end{pmatrix}, (t_1, t_2) \right) \mid b \in \mathbb{K}, (t_1, t_2) \in (\mathbb{K}^*)^2 \right\}.$$

We have a $\mathbb{G}_a \rtimes_{\theta} G_e$ -isomorphism $X \simeq (\mathbb{G}_a \rtimes_{\theta} G_e)/H_1$. Similarly, for $\theta = (1, 1)$ the isotropy subgroup of $(1, 0, 0, 1)$ is

$$H_2 = \left\{ \left(-ct_1^{-1}, \begin{pmatrix} t_1 & 0 \\ c & t_1^{-1} \end{pmatrix}, (t_1, t_2) \right) \mid c \in \mathbb{K}, (t_1, t_2) \in (\mathbb{K}^*)^2 \right\},$$

and $X \simeq (\mathbb{G}_a \rtimes_{\theta} G_e)/H_2$.

The next two lemmas will be used in the proof of Theorem 4.18.

Lemma 4.16. *Let X be an affine G_e -spherical variety of type I. Assume that the torus \mathbb{T} acts faithfully on X . Then every exterior normalized \mathbb{G}_a -action on X is vertical for the induced \mathbb{T} -action.*

Proof. According to Theorem 4.5, it is enough to consider the $\mathbb{G}_a \rtimes_{\theta} G_e$ -homogeneous variety X_{ρ} , where ρ is the distinguished ray of $\theta \in \text{Rt}_{\text{ext}}(X)$. Note that a priori X_{ρ} is not necessarily affine. By Theorem 4.5(ii) and Lemma 3.10, we have $\rho \in \text{lin}(\mathcal{V}) = e^{\perp}$. This implies that X_{ρ} is affine and described by a σ -polyhedral divisor over \mathbb{A}^1 , where $\sigma = \mathbb{Q}_{\geq 0} \cdot \rho$. By [Lie10a, Lemma 3.1], the equality $\rho(\theta) = -1$ implies that the \mathbb{G}_a -action on X_{ρ} is vertical. \square

Lemma 4.17. *Let $\varphi \in \mathbb{K}(t)^*$, $s \in \mathbb{Z}$, $d \in \mathbb{Z} \setminus \{0\}$, $v \in N_{\mathbb{Q}}$, $\rho \in N_{\mathbb{Q}} \setminus \{0\}$, and $e, \theta \in M$. Consider the derivations $\partial, \partial_{\bullet}$ on $\mathbb{K}(t)[M]$ defined by the relations*

$$\begin{aligned} \partial_{\bullet}(t^r \chi^m) &= d \cdot (v(m) + r) \chi^{m+e} t^{r+s}, \\ \partial(t^r \chi^m) &= \rho(m) \varphi t^r \chi^{m+\theta} \text{ for all } (m, r) \in M \oplus \mathbb{Z}. \end{aligned}$$

Then $[\partial, \partial_{\bullet}] = 0$ if and only if $\rho \in e^{\perp}$, $v(\theta) \in \mathbb{Z}$, and $\varphi = \lambda t^{-v(\theta)}$ for some $\lambda \in \mathbb{K}^$.*

Proof. Let us compute the Lie bracket $[\partial_{\bullet}, \partial](t^r \chi^m)$. For all $(m, r) \in M \oplus \mathbb{Z}$, we have

$$\begin{aligned} \partial_{\bullet} \circ \partial(t^r \chi^m) &= d \cdot \rho(m) \left[\frac{d\varphi}{dt} t + (v(m + \theta) + r) \varphi \right] t^{r+s} \chi^{m+e+\theta}, \text{ and} \\ \partial \circ \partial_{\bullet}(t^r \chi^m) &= d \cdot \rho(m + e) (v(m) + r) \varphi t^{r+s} \chi^{m+e+\theta}. \end{aligned}$$

Thus, the condition $[\partial, \partial_{\bullet}](t^r \chi^m) = 0$ is equivalent to

$$\rho(m) \left(v(\theta) \varphi + t \frac{d\varphi}{dt} \right) = \rho(e) (v(m) + r) \varphi$$

for any $r \in \mathbb{Z}$, $m \in M$. The left-hand side does not depend on r , hence $[\partial, \partial_{\bullet}] = 0$ implies $\rho(e)r\varphi = 0$ for all $r \in \mathbb{Z}$ and is equivalent to

$$\begin{cases} \rho(e) = 0, \\ v(\theta) \varphi + t \frac{d\varphi}{dt} = 0. \end{cases}$$

The latter equation is equivalent to $\varphi = \lambda t^{-v(\theta)}$ for some $\lambda \in \mathbb{K}^*$. \square

The following result gives a description of Demazure roots of an affine G_e -spherical variety.

Theorem 4.18. *Let X be an affine G_e -spherical variety. Assume that $e \neq 0$ and that \mathbb{T} acts faithfully on X . Let $(\mathcal{C}, \mathcal{F})$ be the associated colored cone of X with respect to the B_{\pm} -action. Let Γ be the polyhedral cone generated by the subset \mathcal{C} and by the images of all colors of X . Then the following hold.*

- (i) If X is of type I, then X is isomorphic to an embedding of G_e/H , where G_e/H is as in 3.5. In the reflexive case,

$$\text{Rt}_{\text{ext}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L \mid \rho_\theta \in \text{lin.}(\mathcal{V}), v_0(\theta) = v_1(\theta) = 0\},$$

and in the skew case

$$\text{Rt}_{\text{ext}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L \mid \rho_\theta \in \text{lin.}(\mathcal{V}), v_1(\theta) = 0\}.$$

Furthermore, if X is skew or v_0, v_1 are linearly dependent, then $\text{Rt}_{\text{int}}(X) = \emptyset$; otherwise, we have

$$\text{Rt}_{\text{int}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L \mid v_0(\theta) = v_1(\theta) = -1 \text{ or } v_0(\theta) = v_1(\theta) = 1\}.$$

- (ii) If X is not of type I, then X is a $\tilde{G}_{\tilde{e}}$ -spherical variety of type II (see Lemma 3.14) isomorphic to an embedding of X_Σ as in Lemma 3.16. We have $\text{Rt}_{\text{int}}(X) = \emptyset$ and, in the notation of 3.18,

$$\text{Rt}_{\text{ext}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L_\pm \mid \theta \in \rho_-^\perp \cap \rho_+^\perp \text{ and } \rho_\theta \in e^\perp\}.$$

Proof. (i) Note that the description of $\text{Rt}_{\text{int}}(X)$ is a direct consequence of Theorem 4.14. Let us describe the exterior Demazure roots of X . Let $\theta \in \text{Rt}_{\text{ext}}(X)$ with the distinguished root $\rho = \rho_\theta$. Then by Lemma 4.16, the corresponding \mathbb{G}_a -action on X_ρ is obtained from a vertical LND ∂ on $\mathbb{K}[X_\rho]$ of degree θ such that the algebra $\mathbb{K}[X]$ is ∂ -stable, see 2.10. By Lemma 4.17, the condition $[\partial, \partial_-] = 0$ implies that $v_0(\theta) \in \mathbb{Z}$ and ∂ is given by the formula

$$\partial(Q\chi^m) = \rho(m)t^{-v_0(\theta)}Q\chi^{m+\theta},$$

where $Q \in \mathbb{K}(t)$ and $m \in M$. The other implication $\rho \in e^\perp$ is equivalent to $\rho \in \text{lin.}(\mathcal{V})$.

Reflexive case. Using formula for ∂_+ in Remark 2.26 and the equality $\rho(e) = 0$, we have

$$[\partial_+, \partial](Q\chi^m) = \rho(m) \left(v_1(\theta) - v_0(\theta) \frac{t-1}{t} \right) t^{-v_0(\theta)} Q\chi^{m+e+\theta} \text{ for any } Q \in \mathbb{K}(t), m \in M.$$

Therefore, ∂ and ∂_+ commute if and only if $v_0(\theta) = v_1(\theta) = 0$. We conclude by remarking that $\partial(\mathbb{K}[X]) \subset \mathbb{K}[X]$ gives the condition $\theta \in \text{Rt}(\Gamma)$.

Skew case. Similarly,

$$[\partial_+, \partial](Q\chi^m) = 2\rho(m)v_1(\theta)t^{-v_0(\theta)}Q\chi^{m+e+\theta} \text{ for any } Q \in \mathbb{K}(t), m \in M.$$

Thus, ∂ and ∂_+ commute if and only if $v_1(\theta) = 0$. Moreover, let \mathfrak{D}' be the polyhedral divisor over \mathbb{A}^1 associated to X_ρ ; here we identify $\mathbb{K}[X_\rho]$ with a subalgebra of $\mathbb{K}[G_e/H]$. Note that the condition on the vertical derivation $\partial = \partial_{\theta, t^{-v_0(\theta)}}$ imposed by 2.10, namely

$$t^{-v_0(\theta)} \in \Phi_\theta^* = H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}([\mathfrak{D}'(\theta)])) \setminus \{0\},$$

is always fulfilled. Again, we have the condition $\theta \in \text{Rt}(\Gamma)$.

- (ii) $\mathbb{K}[X]$ is the semigroup algebra $\mathbb{K}[\sigma^\vee \cap M]$, where $\sigma^\vee \subset M_{\mathbb{Q}}$ is the weight cone for the \mathbb{T} -action. Let $\theta \in \text{Rt}(X)$ with the distinguished root $\rho = \rho_\theta$. Then $\theta \in \text{Rt}(\sigma)$ and it corresponds to the homogeneous LND ∂ given by

$$\partial(\chi^m) = \rho(m)\chi^{m+\theta}.$$

The conditions $[\partial, \partial_-] = [\partial, \partial_+] = 0$ are equivalent to (see the notation of 3.15)

$$\begin{cases} \rho_-(\theta)\rho(m) = -\rho(e)\rho_-(m) \text{ for any } m \in \sigma^\vee \cap M, \\ \rho_+(\theta)\rho(m) = \rho(e)\rho_+(m) \text{ for any } m \in \sigma^\vee \cap M. \end{cases}$$

In other words, $\rho_-(\theta)\rho = -\rho(e)\rho_-$ and $\rho_+(\theta)\rho = \rho(e)\rho_+$. Moreover, these two vectors are collinear. Since ρ_- and ρ_+ are not, $\rho(e) = 0$. Moreover, $\rho \neq 0$ implies $\rho_-(\theta) = \rho_+(\theta) = 0$. But $\rho(\theta) = -1$ by definition, hence $\rho \notin \{\rho_-, \rho_+\}$. So, Lemma 3.16(iii) implies that ρ is exterior and $\text{Rt}_{\text{int}}(X) = \emptyset$. Finally, under the conditions $\rho(e) = \rho_-(\theta) = \rho_+(\theta) = 0$ the induced \mathbb{G}_a -action is indeed normalized by G_e . \square

Corollary 4.19. *Let $e \in M$ be a nonzero element. Then the answer to Question 4.11 is affirmative for the reductive group G_e .*

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